Phasors and Complex Numbers

1. Notation for instantenious and phasor quantites:

(a). Euler's Identity: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$

$\mathrm{Re}\left(e^{j\cdot\theta}\right) = \cos\theta$ and $\mathrm{Im}\left(e^{j\cdot\theta}\right) = \sin\theta$

(b). Instantenious: $v(t) = V_m\cdot\cos(\omega t + \phi_v)$ and $i(t) = I_m\cdot\cos(\omega t + \phi_i)$

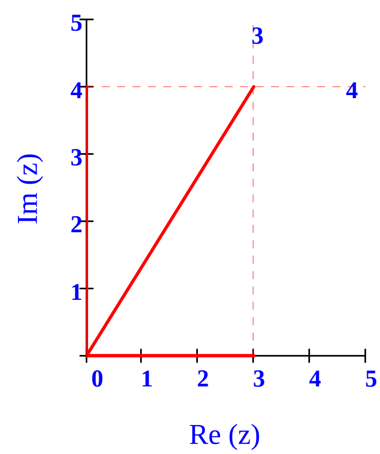
(c). RMS Phasor Quantity: $\underline{V} = \left(\frac{V_m}{\sqrt{2}}\right)\angle\,\phi_v = V\angle\,\phi_v$ and $\underline{I} = \left(\frac{I_m}{\sqrt{2}}\right)\angle\,\phi_v = I\angle\,\phi_v$

2. Complex numbers:

By complex number in canonical (algebraical) form, we understand a number of the form $\underline{z} = x + jy$, where x and y are real numbers and j is imaginary unit defined by $j^2 = -1$. The real number x is called the real component or real part of the complex number $\underline{z}$, or $\mathrm{Re}(\underline{z})$. The real number y is called the imaginary component or imaginary part of the xomplex number $\underline{z}$, or $\mathrm{Im}(\underline{z})$. The complex number (i.e. phasor) can be geometrically represented on the complex plane, z-plane, or R-X plane. The vector that represents $\underline{z}$ number, has a lenght $r = |\underline{z}| = \sqrt{x^2 + y^2}$, and its direction angledefined for $\underline{z} \neq 0$, via $\cos(\phi) = \frac{x}{|\underline{z}|}$, $\sin(\phi) = \frac{y}{|\underline{z}|}$. The lenght is called also **absolute value** or **modulus** of $\underline{z}$ (written $\mathrm{mod}(\underline{z})$). The ϕ is called a argument of $\underline{z}$. Complex number has: main argument (i.e $\arg(\underline{z})$)when argument belongs to $(-\pi, \pi)$. Set of all arguments of the complex number $\underline{z}$, (i.e. $\mathrm{Arg}(\underline{z})$) defined as:

$\mathrm{Arg}(\underline{z}) = \arg(\underline{z}) + 2\cdot\pi\cdot k$, (k = +/-1, +/-2, +/-3, ...).

Following are properties of the complex numbers in canonical form:



$$z = |z|\left(\cos(\phi) + j\cdot\sin(\phi)\right)$$

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

$$(a + jb)\cdot(c + jd) = (a\cdot c - b\cdot d) + j(b\cdot c + a\cdot d)$$

$$\frac{(a + jb)}{(c + jd)} = \frac{a\cdot c + b\cdot d}{c^2 + d^2} + j\cdot\frac{b\cdot c - a\cdot d}{c^2 + d^2}$$

$$z_1\cdot z_2 = \left[r_1\left(\cos(\phi_1) + j\cdot\sin(\phi_1)\right)\right]\cdot r_2\left(\cos(\phi_2) + j\cdot\sin(\phi_2)\right)$$

$$z_1\cdot z_2 = (r_1 r_2)\left[\cos(\phi_1 + \Phi_2) + j\cdot\sin\cdot(\phi_1 + \phi_2)\right]$$

$$\frac{z_1}{z_2} = \frac{\left[r_1\left(\cos(\phi_1) + j\cdot\sin(\phi_1)\right)\right]}{\left[r_2\left(\cos(\phi_2) + j\cdot\sin(\phi_2)\right)\right]} = \frac{r_1}{r_2}\cdot\left(\cos(\phi_1 - \phi_2) + j\cdot\sin(\phi_1 - \phi_2\right.$$

$\frac{1}{z} = \frac{1}{r}\cdot\left(\cos(-\phi) + j\cdot\sin(-\phi)\right)$. Conjugate is $\bar{\underline{z}} = x - jy$,

Following are properties of the complex numbers in polar form: $\underline{z} = r\cdot e^{j\phi}$

$$z_1\cdot z_2 = \left(r_1 e^{j\phi_1}\right)\cdot\left(r_2 e^{j\phi_2}\right) = r_1\cdot r_2\cdot e^{j(\phi_1+\phi_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\phi_1}}{r_2 e^{j\phi_2}} = \frac{r_1}{r_2}\cdot e^{j(j(\phi_1-\phi_2)}$$

Per Unit Calculations

DEFINITION OF A PER UNIT OF QUATITY

$$(\text{per unit value}) = \left(\frac{\text{actual value}}{\text{base value}}\right)$$

BASE POWER EQUATIONS

$$(\text{base VA}_{3ph}) = 1000 \cdot (\text{base kVA}_{3ph}) = 10^6 \cdot (\text{base MVA}_{3ph})$$

BASE VOLTAGE EQUATION

$$(\text{baseV}_{LL}) = 1000 \cdot (\text{base kV}_{LL})$$

BASE CURRENT EQUATIONS

$$(\text{base A}) = \frac{(\text{base VA}_{3ph})}{\sqrt{3} \cdot (\text{base V}_{LL})} = \frac{(\text{base kVA}_{3ph})}{\sqrt{3} \cdot (\text{base kV}_{LL})} = \frac{1000(\text{base MVA}_{3ph})}{\sqrt{3} \cdot (\text{base kV}_{LL})}$$

BASE IMPEDANCE / ADMITANCE EQUATIONS

$$(\text{base } \Omega) = \frac{(\text{base V}_{LL})}{\sqrt{3} \cdot (\text{base A})} = \frac{1000 \cdot (\text{base kV}_{LL})}{\sqrt{3} \cdot (\text{base A})} = \frac{1000 \cdot (\text{base kV}_{LL})^2}{(\text{base kVA}_{3ph})} = \frac{(\text{base kV}_{LL})^2}{(\text{base MVA}_{3ph})}$$

$$(\text{base } \Omega^{-1}) = (\text{base S}) = \frac{1}{(\text{base } \Omega)}$$

CHANGING THE BASE OF A PER UNIT QUANTITY

$$S_{pu.NEW} = S_{pu.OLD} \cdot \frac{(\text{base VA}_{OLD})}{(\text{base VA}_{NEW})}$$

$$V_{pu.NEW} = V_{pu.OLD} \cdot \frac{(\text{base V}_{OLD})}{(\text{base V}_{NEW})}$$

$$I_{pu.NEW} = I_{pu.OLD} \cdot \frac{(\text{base I}_{OLD})}{(\text{base I}_{NEW})} = I_{pu.OLD} \cdot \frac{(\text{base VA}_{OLD})}{(\text{base VA}_{NEW})} \cdot \frac{(\text{base V}_{NEW})}{(\text{base V}_{OLD})}$$

$$Z_{pu.NEW} = Z_{pu.OLD} \cdot \frac{(\text{base Z}_{OLD})}{(\text{base.Z}_{NEW})} = Z_{pu.OLD} \cdot \frac{(\text{base VA}_{NEW})}{(\text{base VA}_{OLD})} \cdot \left[\frac{(\text{base V}_{OLD})}{(\text{base V}_{NEW})}\right]^2$$

Symmetrical Components

$\underline{I_A} = I_A \angle \delta_A$ -> $\underline{I_{A0}} = I_0 \angle \delta_0$ $\underline{I_{A1}} = I_1 \angle \delta_1$ $\underline{I_{A2}} = I_2 \angle \delta_2$

$\underline{I_B} = I_B \angle \delta_B$ $\underline{I_{B0}} = I_0 \angle \delta_0$ $\underline{I_{B1}} = I_1 \angle (\delta_1 - 120deg)$ $\underline{I_{B2}} = I_2 \angle (\delta_2 + 120deg)$

$\underline{I_C} = I_C \angle \delta_C$ $\underline{I_{C0}} = I_0 \angle \delta_0$ $\underline{I_{C1}} = I_1 \angle (\delta_1 + 120deg)$ $\underline{I_{C2}} = I_2 \angle (\delta_2 - 120deg)$

$\underline{I_{A0}}, \underline{I_{B0}}, \underline{I_{C0}}$ - are called the zero sequence components

$\underline{I_{A1}}, \underline{I_{B1}}, \underline{I_{C1}}$ - are called the positive sequence components

$\underline{I_{A2}}, \underline{I_{B2}}, \underline{I_{C2}}$ - are called the negative sequence components

Note:
- All three sets of symmetrical componnets are balanced (magnituteds equal)
- Only the positive and negative sequence componets are symmetric (equally spaced over 0 - 360deg)
- The negative sequence components are in the reverse order from the phase sequence ABC

The unbalanced phasors are then defined to be linear combination of the symmetrical components:

$$\underline{I_A} = \underline{I_{A0}} + \underline{I_{A1}} + \underline{I_{A2}}$$

$$\underline{I_B} = \underline{I_{B0}} + \underline{I_{B1}} + \underline{I_{B2}}$$

$$\underline{I_C} = \underline{I_{C0}} + \underline{I_{C1}} + \underline{I_{C2}}$$

If the special complex constant a is defined, then the above equations can be simplifed:

If $a = 1 \angle (120deg)$ then $a^2 = [1 \angle (120deg)] \cdot [1 \angle (120deg)] = 1 \angle (240deg) = 1 \angle (-120deg)$

$\underline{I_{A0}} = I_0 \angle \delta_0 = \underline{I_0}$ $\underline{I_{A1}} = I_1 \angle \delta_1 = \underline{I_1}$ $\underline{I_{A2}} = I_2 \angle \delta_2 = \underline{I_2}$

$\underline{I_{B0}} = I_0 \angle \delta_0 = \underline{I_0}$ $\underline{I_{B1}} = I_1 \angle (\delta_1 - 120deg) = a^2 \cdot \underline{I_1}$ $\underline{I_{B2}} = I_2 \angle (\delta_2 + 120deg) = a \cdot \underline{I_2}$

$\underline{I_{C0}} = I_0 \angle \delta_0 = \underline{I_0}$ $\underline{I_{C1}} = I_1 \angle (\delta_1 + 120deg) = a \cdot \underline{I_1}$ $\underline{I_{C2}} = I_2 \angle (\delta_2 - 120deg) = a^2 \cdot \underline{I_2}$

$$\underline{I_A} = \underline{I_{A0}} + \underline{I_{A1}} + \underline{I_{A2}} = \underline{I_0} + \underline{I_1} + \underline{I_2}$$

$$\underline{I_B} = \underline{I_{B0}} + \underline{I_{B1}} + \underline{I_{B2}} = \underline{I_0} + a^2 \cdot \underline{I_1} + a \cdot \underline{I_2}$$

$$\underline{I_C} = \underline{I_{C0}} + \underline{I_{C1}} + \underline{I_{C2}} = \underline{I_0} + a \cdot \underline{I_1} + a^2 \cdot \underline{I_2}$$

Putting these equation into matrix form will allow the calulation of the symmetrical components from the unbalanced set of phasors:

$$\begin{pmatrix} \underline{I_A} \\ \underline{I_B} \\ \underline{I_C} \end{pmatrix} = \frac{1}{h} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \underline{a}^2 & \underline{a} \\ 1 & \underline{a} & \underline{a}^2 \end{pmatrix} \cdot \begin{pmatrix} \underline{I_0} \\ \underline{I_1} \\ \underline{I_2} \end{pmatrix} \qquad \begin{pmatrix} \underline{I_0} \\ \underline{I_1} \\ \underline{I_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \underline{a}^2 & \underline{a} \\ 1 & \underline{a} & \underline{a}^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \underline{I_A} \\ \underline{I_B} \\ \underline{I_C} \end{pmatrix} = \frac{h}{3} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \underline{a} & \underline{a}^2 \\ 1 & \underline{a}^2 & \underline{a} \end{pmatrix} \cdot \begin{pmatrix} \underline{I_A} \\ \underline{I_B} \\ \underline{I_C} \end{pmatrix}$$

Note: $h = 1$ for the Fortescue transformation (this presentation)

$h = \sqrt{3}$ for the power invariant transformation

The same transfromation equations are used for L-N voltages also.

In general, for n phase system, there are $i = n - 1$ symmetrical compnents and one "zero" component operators. The component operator λ_i is defined as $\lambda_i = e^{j\cdot\frac{2\cdot\pi}{n}\cdot(i-1)}$. For 3-phase system with $n = 3$, there are $n - 1 = 2$ componetns (positive and negative) $\lambda_2 = e^{j\cdot\frac{2\cdot\pi}{3}\cdot(2-1)} = e^{j\cdot\frac{2\cdot\pi}{3}} = e^{j\cdot 120deg} = a$

rectangular notation	polar notation
$a := -\frac{1}{2} + j\cdot\frac{\sqrt{3}}{2} = -0.5 + 0.866i$	$z2r\theta(a) = $ "(1∡ 120°)"
$a^2 = -0.5 - 0.866i$	$z2r\theta(a^2) = $ "(1∡ -120°)"
$a^3 = 1$	$z2r\theta(a^3) = $ "(1∡ 0°)"
$a^4 = -0.5 + 0.866i$	$z2r\theta(a^4) = $ "(1∡ 120°)"
$a^5 = -0.5 - 0.866i$	$z2r\theta(a^5) = $ "(1∡ -120°)"
$a + a^2 + 1 = 0$	
$a + a^2 = -1$	$z2r\theta(a + a^2) = $ "(1∡ 180°)"
$a - a^2 = 1.732i$	$z2r\theta(a - a^2) = $ "(1.7321∡ 90°)"
$a^2 - a = -1.732i$	$z2r\theta(a^2 - a) = $ "(1.7321∡ -90°)"
$1 - a = 1.5 - 0.866i$	$z2r\theta(1 - a) = $ "(1.7321∡ -30°)"
$1 - a^2 = 1.5 + 0.866i$	$z2r\theta(1 - a^2) = $ "(1.7321∡ 30°)"
$a - 1 = -1.5 + 0.866i$	$z2r\theta(a - 1) = $ "(1.7321∡ 150°)"
$a^2 - 1 = -1.5 - 0.866i$	$z2r\theta(a^2 - 1) = $ "(1.7321∡ -150°)"
$1 + a = 0.5 + 0.866i$	$z2r\theta(1 + a) = $ "(1∡ 60°)"
$1 + a^2 = 0.5 - 0.866i$	$z2r\theta(1 + a^2) = $ "(1∡ -60°)"