2012 EE301 Midterm I Solutions

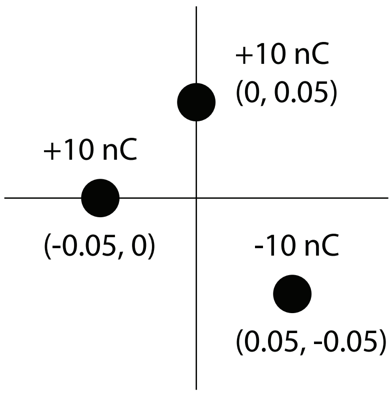
1. Point charges

Three charges are arranged as shown below. All distances are in meters.

a) Determine the electric field vector at the origin.

b) Determine the electric potential at the origin.

c) If another charge of +1 nC is placed at the origin, what force acts on this charge and what is the potential energy of the charge?



Answer:

a) The problem is solved by adding together the electric field vectors produced by each charge separately. For the charge at $(-0.05, 0)$, $\vec{r}' = -0.05\vec{a}_x$ m and $\vec{r} = 0$.

$$\vec{E}_1 = \frac{10 \text{ nC}}{4\pi\epsilon_0} \frac{(0.05\vec{a}_x \text{ m})}{(0.05 \text{ m})^3} \approx (3.6 \times 10^4 \text{ V/m})\vec{a}_x$$

For the charge at $(0, 0.05)$, $\vec{r}' = 0.05\vec{a}_y$ m and $\vec{r} = 0$.

$$\vec{E}_2 = \frac{10 \text{ nC}}{4\pi\epsilon_0} \frac{(-0.05\vec{a}_y \text{ m})}{(0.05 \text{ m})^3} \approx (-3.6 \times 10^4 \text{ V/m})\vec{a}_y$$

For the charge at $(0.5, -0.5)$, $\vec{r}' = 0.05\vec{a}_x - 0.05\vec{a}_y$ m and $\vec{r} = 0$.

$$\vec{E}_3 = \frac{-10 \text{ nC}}{4\pi\epsilon_0} \frac{(-0.05\vec{a}_x + 0.05\vec{a}_y \text{ m})}{((0.05 \text{ m})^2 + (0.05 \text{ m})^2)^{3/2}} \approx (1.27 \times 10^4 \text{ V/m})(\vec{a}_x - \vec{a}_y)$$

The total electric field at the origin is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 = (4.9 \times 10^4 \ \vec{a}_x - 4.9 \times 10^4 \ \vec{a}_y) \text{ V/m}$$

b) Notice that the +10 nC charge at $(-0.05, 0)$ produces the same electric potential at the origin as the 10 nC charge at $(0, 0.05)$ which is

$$V_1 = \frac{10 \text{ nC}}{4\pi\epsilon_0(0.05 \text{ m})} = 1.80 \times 10^3 \text{ V}$$

The potential of the -10 nC charge at $(0.05, -0.05)$ at the origin is

$$V_3 = \frac{-10\ \text{nC}}{4\pi\epsilon_0(((0.05\ \text{m})^2 + (0.05\ \text{m})^2)^{1/2})} = -1.27 \times 10^3\ \text{V}$$

The total potential at the origin is

$$V = 2V_1 + V_3 = 2.32 \times 10^3\ \text{V}$$

c) The force on the extra charge is

$$\vec{F} = Q\vec{E} = (4.9 \times 10^{-5}\ \vec{a}_x - 4.9 \times 10^{-5}\ \vec{a}_y)\ \text{N}$$

and the energy is

$$W = QV = 2.32 \times 10^{-6}\ \text{J}$$

2. Point charges

a) A charge Q is located at the center of a cube with side a. What is the total electric flux that passes through the walls of the cube?

b) A charge of 1 nC is 1 cm above an infinite, grounded metal plane. What is the magnitude and direction of the force acting on the charge?

Answer:

a) The cube is a closed surface so by Gauss's law the total flux through the cube is just Q.

b) By the method of image charges, a charge located a distance d above a grounded metal plane is equivalent to a dipole, that is two charges, $+Q$ and $-Q$, separated by $2d$. Therefore the force on the charge is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{R^2}\vec{a}_R = \frac{1}{4\pi\epsilon_0}\frac{(1\ \text{nC})(-1\ \text{nC})}{(0.02\ \text{m})^2}\vec{a}_z = -2.25 \times 10^{-5}\ \text{N}\,\vec{a}_z$$

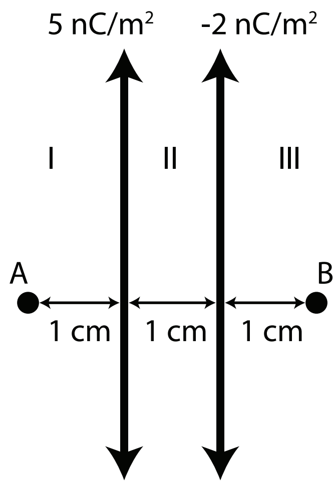
Notice that the force is always directed toward the metal regardless of the sign of the charge.

3. Planes of charge

Two infinite planes of charge are parallel to each other as shown below (viewed end on) and separated by 1 cm. The left plane has surface charge density $\rho_S = 5\ \text{nC/m}^2$ and the right plane has surface charge density $\rho_S = -2\ \text{nC/m}^2$.

a) Determine the electric field in the three regions I, II, III.

b) What is the potential difference between point A in region I and point B in region III? Point A is 1 cm to the left of the left plane of charge, and point B is 1 cm to the right of the right plane of charge.



Answer:

a) For a plane of charge, $|\vec{E}| = \rho_S/(2\epsilon_0)$; the field points away from a plane of positive charge and toward one of negative charge. If you do not remember this equation then it can be easily derived using Gauss's law. The total electric field in each section is the superposition of the fields produced by each plane. Let the z-axis point to the right.

$$\text{I} \quad \vec{E} = -\frac{5 \times 10^{-9}\ \text{C/m}^2}{2\epsilon_0}\vec{a}_z + \frac{2 \times 10^{-9}\ \text{C/m}^2}{2\epsilon_0}\vec{a}_z = -169\ \text{V/m}\,\vec{a}_z$$

$$\text{II} \quad \vec{E} = \frac{5 \times 10^{-9}\ \text{C/m}^2}{2\epsilon_0}\vec{a}_z + \frac{2 \times 10^{-9}\ \text{C/m}^2}{2\epsilon_0}\vec{a}_z = 395\ \text{V/m}\,\vec{a}_z$$

$$\text{III} \quad \vec{E} = \frac{5 \times 10^{-9}\ \text{C/m}^2}{2\epsilon_0}\vec{a}_z - \frac{2 \times 10^{-9}\ \text{C/m}^2}{2\epsilon_0}\vec{a}_z = 169\ \text{V/m}\,\vec{a}_z$$

b) Take the integration path to be a straight line between A and B.

$$V_{AB} = -\int_B^A \vec{E} \cdot \vec{dl}$$

The integral divides into three pieces, one piece for each region. Since the electric field is a constant in a given region, the integral evaluates to El where l is the length of the path in that region. Thus

$$V_A - V_B = (-169\ \text{V/m})(0.01\ \text{m}) + (395\ \text{V/m})(0.01\ \text{m}) + (169\ \text{V/m})(0.01\ \text{m}) = 3.95\ \text{V}$$

4. Gauss's law

Space is filled with a volume charge density that varies according to the formula (spherical coordinates)

$$\rho_V = \frac{C}{r^2}$$

where C is a constant.

a) What information can be obtained about the electric field $\vec{E}(r, \theta, \phi)$ from the symmetries of the charge distribution? (You do not need to write down the reasoning, only the final result.)

b) Use Gauss's Law to calculate the electric field everywhere.

Answer:

a) By symmetry, $\vec{E}(r) = E_r \vec{a}_r$; e.g. the electric field only has a radial component and only depends on the r variable.

b) Use a sphere with radius R for the Gaussian surface S in the flux integral. The flux is

$$\Phi = \epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} E_r \vec{a}_r \cdot R^2 \sin\theta \, d\theta d\phi \, \vec{a}_r = \epsilon_0 E_r 4\pi R^2$$

The charge enclosed by S is

$$Q_{\text{encl}} = \int_V \rho_V \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{C}{r^2} r^2 \sin\theta \, d\theta d\phi = 4\pi C R$$

Equating to the flux gives

$$E_r = \frac{C}{\epsilon_0 R}$$

But the radius R is arbitrary and might as well be called the variable r. So

$$\vec{E} = \frac{C}{\epsilon_0 r} \vec{a}_r$$

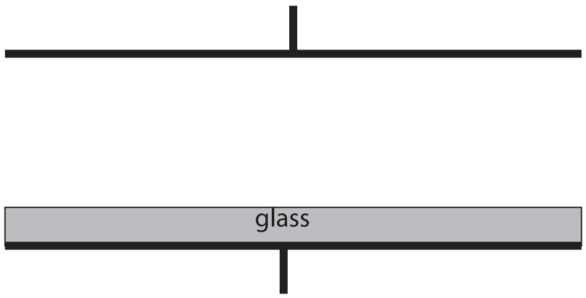
5. Capacitor

A parallel-plate capacitor has plate area $A = 50$ cm^2 and plate separation $d = 5$ mm. It is filled with dry air ($\epsilon_R = 1$) with a dielectric breakdown strength of 3×10^6 V/m (i.e. if the electric field exceeds the breakdown strength then a spark or corona discharge will occur).

a) At what voltage applied to the capacitor will breakdown occur? At this maximum voltage, how much energy is stored in the capacitor?

b) A thin piece of glass is now inserted into the capacitor as shown below. The glass is 1 mm thick, has a dielectric constant $\epsilon_R = 6$ and a large dielectric breakdown strength of

3×10^7 V/m. At what voltage applied to the capacitor will breakdown occur? Comment on why the breakdown voltage has changed from the value in part a. At this maximum voltage, how much energy is stored in the capacitor?



Answer:

a) The magnitude of the electric field in the capacitor is simply V/d. Therefore the maximum voltage is

$$V_{\max} = E_{\text{br}} d = (3 \times 10^6 \text{ V/m})(5 \times 10^{-3} \text{ m}) = 15{,}000 \text{ V}$$

Since the electric field is constant in the capacitor, the energy is simply

$$W = \epsilon_0 E_{\text{br}}^2 A d/2 = 1.0 \times 10^{-3} \text{ J}$$

b) The displacement field in the capacitor is $\vec{D} = \rho_S \vec{a}_z = (Q/A)\vec{a}_z$. If you do not remember this formula then it can be easily derived using Gauss's law. The electric field in the air and the glass is

$$\vec{E}_{\text{air}} = \frac{Q}{A\epsilon_0}\vec{a}_z \qquad \vec{E}_{\text{glass}} = \frac{Q}{A\epsilon_0\epsilon_R}\vec{a}_z = \frac{\vec{E}_{\text{air}}}{\epsilon_R}$$

The potential is determined by integrating $\vec{E}$ over a path connecting the two plates.

$$V = E_{\text{air}} d_{\text{air}} + E_{\text{glass}} d_{\text{glass}} = E_{\text{air}} \left(d_{\text{air}} + \frac{d_{\text{glass}}}{\epsilon_R} \right)$$

The glass has such a large dielectric breakdown strength that it will not breakdown. However, the air will still breakdown when the field reaches E_{br}. Therefore,

$$V_{\max} = E_{\text{br}} \left(d_{\text{air}} + \frac{d_{\text{glass}}}{\epsilon_R} \right) = 12{,}500 \text{ V}$$

Notice that inserting the glass has reduced the maximum voltage. The reason for this somewhat counterintuitive result is that a dielectric reduces the electric field within it. For a given voltage applied to the capacitor, the lower field inside the glass must be compensated by a greater field in the air. By enhancing the field in the air, less voltage is required to reach breakdown.

When the field in the air is $E_{\rm br}$, the field in the glass is $E_{\rm br}/\epsilon_R$. The energy stored at the new maximum voltage is

$$W = \epsilon_0 E_{\rm air}^2 A d_{\rm air}/2 + \epsilon_0 \epsilon_R E_{\rm glass}^2 A d_{\rm glass}/2 = 8.3 \times 10^{-4}\ \mathrm{J}$$