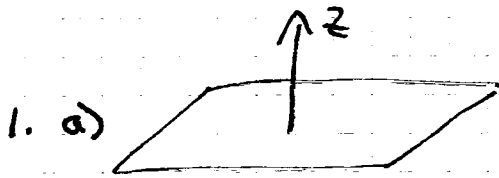


Midterm Solutions

①



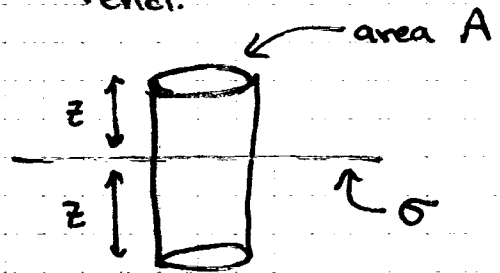
Symmetries

translation in x & y \Rightarrow no dependence on x, y
 rotation about z \rightarrow no E_x or E_y components.

Gauss's Law

$$\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = Q_{\text{encl.}}$$

Choose S as shown



Since $E = E_z \vec{a}_z$

$\vec{E} \cdot d\vec{S} = 0$ for the sidewall and

$\vec{E} \cdot d\vec{S} = |E_z| ds$ for the end caps E_z constant.

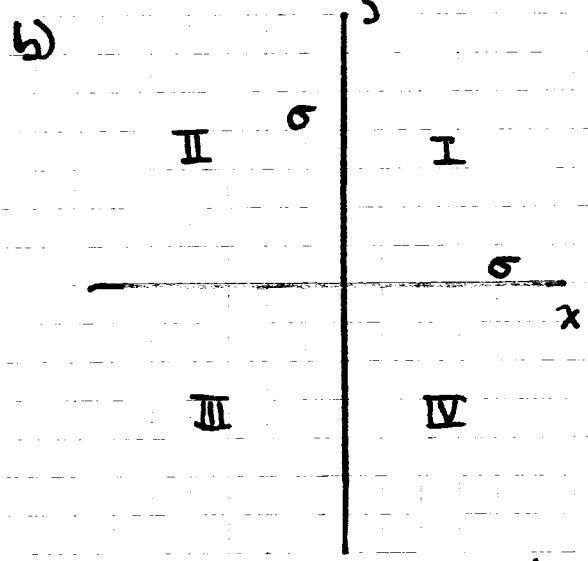
Also $|E_z|$ is the same for the top and bottom endcap because of symmetry.

$$\text{so } \epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = 2A E_z \epsilon_0$$

$$Q_{\text{encl.}} = A\sigma$$

$$\text{so } E_z = \frac{\sigma}{2\epsilon_0}$$

$$\vec{E} = \pm \frac{\sigma}{2\epsilon_0} \vec{a}_z \quad \begin{array}{l} + \text{ for } z > 0 \\ - \text{ for } z < 0 \end{array}$$



Use super position

The field from the sheet in the x-z plane is

$$\vec{E} = \pm \frac{\sigma}{2\epsilon_0} \vec{a}_y \quad \begin{array}{l} + \quad y > 0 \\ - \quad y < 0 \end{array}$$

The field from the sheet in the y-z plane is

$$\vec{E} = \pm \frac{\sigma}{2\epsilon_0} \vec{a}_x \quad \begin{array}{l} + \quad x > 0 \\ - \quad x < 0 \end{array}$$

so in

Region I $\vec{E} = \frac{\sigma}{2\epsilon_0} (\vec{a}_x + \vec{a}_y)$

Region II $\vec{E} = \frac{\sigma}{2\epsilon_0} (-\vec{a}_x + \vec{a}_y)$

Region III $\vec{E} = \frac{\sigma}{2\epsilon_0} (-\vec{a}_x - \vec{a}_y)$

Region IV $\vec{E} = \frac{\sigma}{2\epsilon_0} (\vec{a}_x - \vec{a}_y)$

c) Again use super position.

For the plane of charge $\vec{E} = \pm \frac{\sigma}{2\epsilon_0} \vec{E}_z \quad \begin{array}{l} + \quad z > 0 \\ - \quad z < 0 \end{array}$

For the sphere

$$\vec{E} = \frac{\frac{4}{3}\pi a^3 \rho}{4\pi\epsilon_0 r^2} \vec{a}_r = \frac{a^3 \rho}{3\epsilon_0 r^2} \vec{a}_r$$

Since a uniform sphere of charge produces the field of a point charge outside the sphere

③

Along the z -axis \vec{a}_r & \vec{a}_z point in the same direction (or opposite directions for $z < 0$)
 so along the z -axis outside the sphere

$$\vec{E} = \pm \left(\frac{\sigma}{2\epsilon_0} + \frac{a^3 \rho}{3\epsilon_0 z^2} \right) \vec{a}_z \quad \begin{array}{l} + \quad z > 0 \\ - \quad z < 0 \end{array}$$

2. $5nC$ @ $(0, 1, 0)$ find \vec{E} @ $(1, 0, 1)$
 $-5nC$ @ $(1, 1, 0)$

$$\vec{r} = \vec{a}_x + \vec{a}_z$$

$$5nC \quad \vec{r}' = \vec{a}_y \quad \vec{r} - \vec{r}' = \vec{a}_x - \vec{a}_y + \vec{a}_z \quad |\vec{r} - \vec{r}'| = \sqrt{3}$$

so

$$\vec{E}_1 = \frac{5nC}{4\pi\epsilon_0} \frac{\vec{a}_x - \vec{a}_y + \vec{a}_z}{3\sqrt{3}}$$

$$-5nC \quad \vec{r}' = \vec{a}_x + \vec{a}_y \quad \vec{r} - \vec{r}' = -\vec{a}_y + \vec{a}_z \quad |\vec{r} - \vec{r}'| = \sqrt{2}$$

$$\vec{E}_2 = \frac{-5nC}{4\pi\epsilon_0} \frac{-\vec{a}_y + \vec{a}_z}{2\sqrt{2}}$$

so

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 = \frac{5nC}{4\pi\epsilon_0} \left(\frac{1}{3\sqrt{3}} \vec{a}_x + \left(\frac{-1}{3\sqrt{3}} + \frac{1}{2\sqrt{2}} \right) \vec{a}_y + \left(\frac{1}{3\sqrt{3}} - \frac{1}{2\sqrt{2}} \right) \vec{a}_z \right)$$

(4)

$$\vec{E}_{\text{total}} \hat{z} = 8.65 \vec{a}_x + 7.24 \vec{a}_y - 7.24 \vec{a}_z \frac{V}{m}$$

3. a) Symmetries

translation along x & $y \Rightarrow$ no dependence on x or y

rotation about axis \parallel to $z \Rightarrow$ no E_x or E_y components.

b) $\rho = A(a^2 - z^2)$ for $-a < z < a$

Symmetries $\Rightarrow \vec{E} = E_z(z) \vec{a}_z$

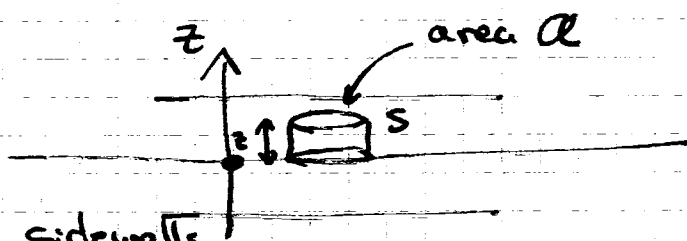
also $\vec{E}(z=0) = 0$

Gauss's Law

$$\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = Q_{\text{encl}}$$

Inside the slab

choose S as shown.



notice $\vec{E} \cdot d\vec{S} = 0$ for sidewalls

and $\vec{E} \cdot d\vec{S} = E_z ds$ for end caps E_z constant

also $E_z = 0$ for bottom end cap

so $\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = \epsilon_0 A E_z$

⑤

The charge enclosed depends on the height of S.

Let the height be z

then

$$Q_{\text{encl.}} = \iiint_0^z A (a^2 - z'^2) dz' dx dy$$

$$= A Q \left(a^2 z' - \frac{1}{3} z'^3 \right) \Big|_0^z$$

so

$$\epsilon_0 Q E_z = A Q \left(a^2 z - \frac{1}{3} z^3 \right)$$

so

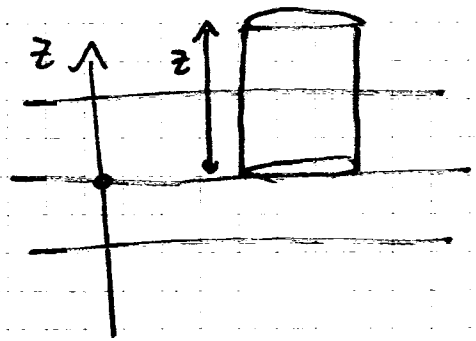
$$\vec{E} = \pm \frac{A}{\epsilon_0} \left(a^2 z - \frac{1}{3} z^3 \right) \vec{a}_z \quad \begin{array}{l} + \quad z > 0 \\ - \quad z < 0 \end{array}$$

Outside the slab

choose S as shown

As above

$$\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = \epsilon_0 Q E_z$$



but now

$$Q_{\text{encl.}} = \iiint_0^a A (a^2 - z'^2) dz' dx dy$$

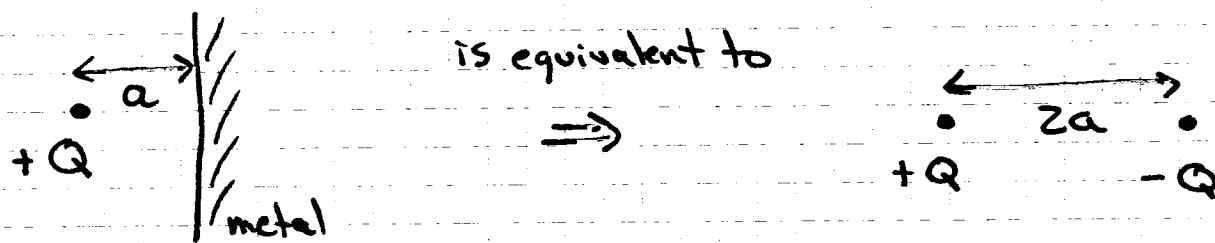
$$= A Q \left(a^2 z' - \frac{1}{3} z'^3 \right) \Big|_0^a = A Q \frac{2}{3} a^3$$

so

$$\vec{E} = \pm \frac{A}{\epsilon_0} \frac{2}{3} a^3 \vec{a}_z \quad \begin{array}{l} + \quad z > 0 \\ - \quad z < 0 \end{array}$$

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4. a) Use the method of image charges

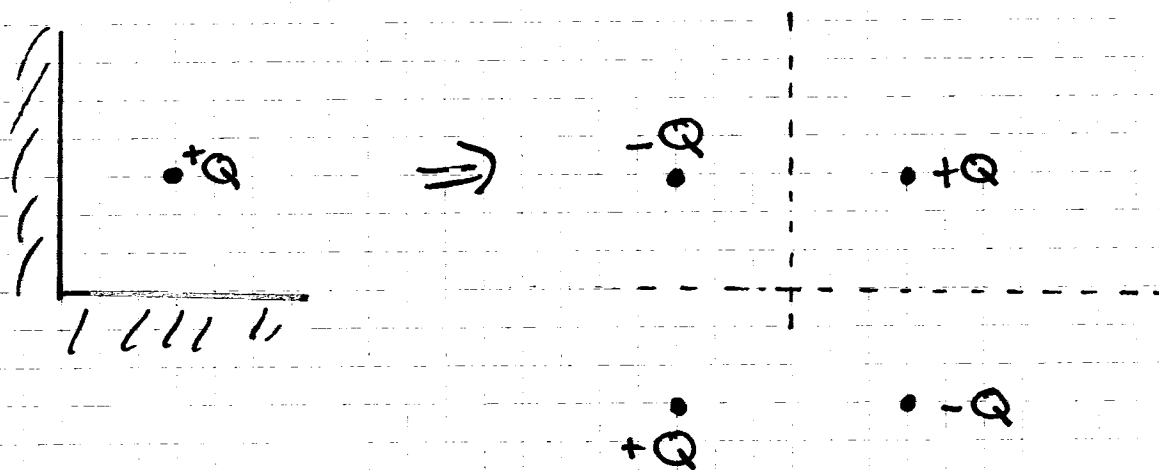


The force on Q is thus $F = \frac{-Q^2}{4\pi\epsilon_0 (2a)^2}$

for $Q = 1\text{ nC}$ and $a = 1\text{ cm}$

$F = 2.25 \times 10^{-5} \text{ Nt}$ attractive to the metal

b) The appropriate image charges are as shown



This arrangement makes both planes of metal equipotentials.

5. The maximum electric field is $E = 1.5 \times 10^6 \frac{V}{cm}$
 $= 1.5 \times 10^8 \frac{V}{m}$

since $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$

so $D = 4.25 \times 10^{-3} \frac{C}{m^2}$

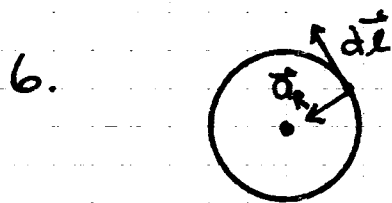
total volume of capacitor

$$V = (25 \times 10^{-6} m) (0.1 m)^2 = 2.5 \times 10^{-7} m^3$$

total energy stored

$$W = \frac{1}{2} \int \vec{E} \cdot \vec{D} dV = \frac{1}{2} (1.5 \times 10^8 \frac{V}{m}) (4.25 \times 10^{-3} \frac{C}{m^2}) \cdot (2.5 \times 10^{-7} m^3)$$

$$W_{max} = 7.97 \times 10^{-2} \text{ Joules.}$$



Biot-Savart $\vec{H} = \frac{1}{4\pi} \oint \frac{I d\vec{l} \times \vec{a}_r}{R^2}$

$$d\vec{l} = a d\varphi \vec{a}_\varphi \quad \vec{a}_r = -\vec{a}_\rho$$

so $d\vec{l} \times \vec{a}_r = a d\varphi \vec{a}_z$ also $R = a$ for the loop.

so $\vec{H} = \frac{I}{4\pi} \int_0^{2\pi} \frac{a d\varphi}{a^2} \vec{a}_z = \frac{I}{2a} \vec{a}_z$