2nd Chance Midterm Solutions.

① 10nC @ (0,0,0) Find $\vec{E}$ at (0,1,1)

-5nC @ (1,0,0)

-5nC @ (1,1,1)

$\vec{r} = \vec{a}_y + \vec{a}_z$

10nC $\vec{r}' = 0$ $\vec{r} - \vec{r}' = \vec{a}_y + \vec{a}_z$ $|\vec{r} - \vec{r}'| = \sqrt{2}$

$$\vec{E} = \frac{10nC}{4\pi\epsilon_0}\left(\frac{\vec{a}_y + \vec{a}_z}{2^{3/2}}\right)$$

-5nC $\vec{r}' = \vec{a}_x$ $\vec{r} - \vec{r}' = -\vec{a}_x + \vec{a}_y + \vec{a}_z$ $|\vec{r} - \vec{r}'| = \sqrt{3}$

$$\vec{E} = \frac{-5nC}{4\pi\epsilon_0}\left(\frac{-\vec{a}_x + \vec{a}_y + \vec{a}_z}{3^{3/2}}\right)$$

-5nC $\vec{r}' = \vec{a}_x + \vec{a}_y + \vec{a}_z$ $\vec{r} - \vec{r}' = -\vec{a}_x$ $|\vec{r} - \vec{r}'| = 1$

$$\vec{E} = \frac{-5nC}{4\pi\epsilon_0}\left(\frac{-\vec{a}_x}{1}\right)$$

so

$$\vec{E}_{tot} = \frac{5nC}{4\pi\epsilon_0}\left[\left(\frac{1}{3^{3/2}} + 1\right)\vec{a}_x + \left(\frac{2}{2^{3/2}} - \frac{1}{3^{3/2}}\right)\vec{a}_y + \left(\frac{2}{2^{3/2}} - \frac{1}{3^{3/2}}\right)\vec{a}_z\right]$$

$$= 45\,V\cdot m\left(1.19\,\vec{a}_x + 0.515\,\vec{a}_y + 0.515\,\vec{a}_z\right)$$

$$= 53.6\,\vec{a}_x + 23.1\,\vec{a}_y + 23.1\,\vec{a}_z \quad \frac{V}{m}$$

The potential is also found through superposition

10nC $\quad V_1 = \dfrac{10nC}{4\pi\epsilon_0} \dfrac{1}{\sqrt{2}}$

-5nC $\quad V_2 = \dfrac{-5nC}{4\pi\epsilon_0} \dfrac{1}{\sqrt{3}}$

-5nC $\quad V_3 = \dfrac{-5nC}{4\pi\epsilon_0} \dfrac{1}{1}$

So $\quad V = \dfrac{5nC}{4\pi\epsilon_0} \left[\dfrac{2}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} - 1 \right]$

so $\quad V = \dfrac{5nC}{4\pi\epsilon_0} (-0.163)$

$= -7.33 \text{ V}$

2. a) $\vec{E}(r,\theta,\varphi) = E_r(r)\,\vec{a}_r$

b) $\rho = \rho_0\,(ar - r^2)$

$$\epsilon_0 \oint_S \vec{E}\cdot d\vec{s} = Q_{encl.}$$

use a spherical surface of radius r for S

$$\Rightarrow \quad \epsilon_0\, 4\pi r^2 E_r = Q_{encl.}$$

outside the sphere $\quad Q_{encl.} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_0 (ar - r^2)\, r^2 \sin\theta \, dr\, d\theta\, d\varphi$

$$\Rightarrow \quad Q_{encl.} = 4\pi\rho_0 \int_0^a ar^3 - r^4 \, dr$$

$$= 4\pi\rho_0 \left(\frac{a}{4} r^4 - \frac{1}{5} r^5 \right)\Big|_0^a$$

$$= 4\pi\rho_0 a^5 \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{4\pi\rho_0 a^5}{20} = \frac{\pi\rho_0 a^5}{5}$$

so $\quad E_r = \dfrac{\rho_0 a^5}{20 r^2 \epsilon_0}$

Inside the sphere

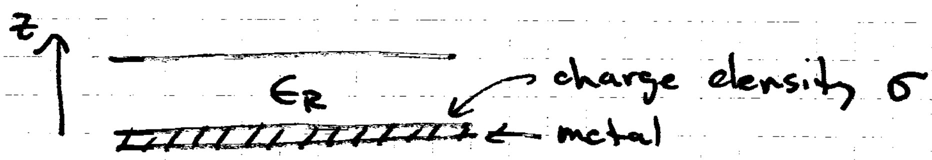
$$Q_{encl.} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0 \left(ar' - r'^2\right) r'^2 \sin\theta \, dr' \, d\theta \, d\varphi$$

$$= 4\pi\rho_0 \left(\frac{a}{4} r'^4 - \frac{1}{5} r'^5 \right)\Big|_0^r$$

$$= 4\pi\rho_0 \left(\frac{a}{4} r^4 - \frac{1}{5} r^5 \right)$$

$$\Rightarrow \quad E_r = \frac{4\pi\rho_0 \left(\frac{a}{4} r^4 - \frac{1}{5} r^5 \right)}{4\pi\epsilon_0 r^2} = \frac{\rho_0}{\epsilon_0}\left(\frac{a}{4} r^2 - \frac{1}{5} r^3 \right)$$

3.

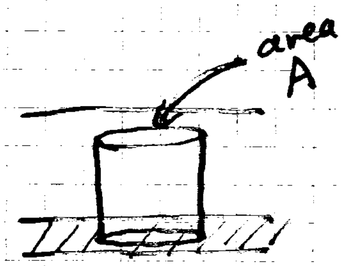


Use Gauss's Law to find $\vec{D}$ which depends only on the free charge σ.

By symmetry $\quad \vec{D} = D_z(z)\, \vec{a}_z$

$$\oint_S \vec{D} \cdot d\vec{s} = Q_{encl.}^{free}$$

choose for S



only contribution to the surface integral comes from the top plate of S

$$\Rightarrow \quad A D_z = A\sigma$$

$$\Rightarrow \quad D_z = \sigma$$

so
$$\vec{D} = \sigma \vec{a}_z$$
This is the displacement field everywhere above the metal

Inside the dielectric

$$\vec{E} = \frac{\vec{D}}{\epsilon_0 \epsilon_R} = \frac{\sigma}{\epsilon_0 \epsilon_R} \vec{a}_z$$

Outside the dielectric

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\sigma}{\epsilon_0} \vec{a}_z$$

The discontinuity in $\vec{E}$ is produced by a surface layer of bound charge at each surface of the dielectric.

since
$$Q^{total} = Q^{free} + Q^{bound}$$

for the surface S
$$Q^{bound}_{encl} = \epsilon_0 \oint_S \vec{E} \cdot d\vec{s} - \oint_S \vec{D} \cdot d\vec{s}$$

$$Q_{encl}^{bound} = \frac{\sigma}{\epsilon_R} A - \sigma A = \sigma A \left(\frac{1}{\epsilon_R} - 1 \right)$$

so the bound charge density on the surface of the dielectric nearest the metal is

$$\sigma_{bound} = \sigma \left(\frac{1}{\epsilon_R} - 1 \right)$$

a corresponding σ_{bound} is on the other surface. but of opposite sign.

4. a) The half plane will intercept 1/4 of the total flux.

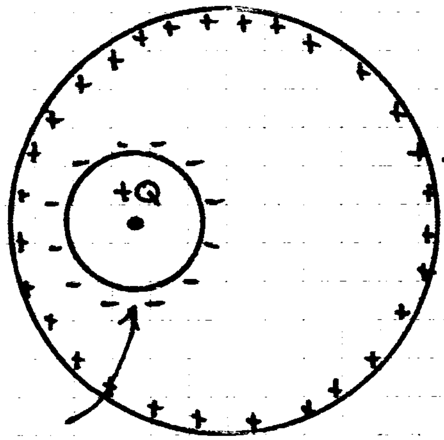
Since the total flux of a point charge is just Q the flux through the 1/2 plane is

$$\frac{100 \mu C}{4} = 25 \mu C$$

b)

air cap.	dielectric cap.	
$\vec{D} = \sigma \vec{a}_z$	$\vec{D} = \sigma \vec{a}_z$	same
$\vec{E} = \frac{\sigma}{\epsilon_0} \vec{a}_z$	$\vec{E} = \frac{\sigma}{\epsilon_0 \epsilon_R} \vec{a}_z$	air cap greater
$V = \frac{\sigma}{\epsilon_0} d$	$V = \frac{\sigma}{\epsilon_0 \epsilon_R} d$	air cap greater

5. a)



uniform + charge on metal's surface total charge +Q

uniform − charge on inner surface of cavity. total charge −Q

Inside Cavity: $\vec{E}$ is just that of the point charge

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\vec{a}_r$$

where the origin is at Q

Inside metal $\qquad \vec{E} = 0$

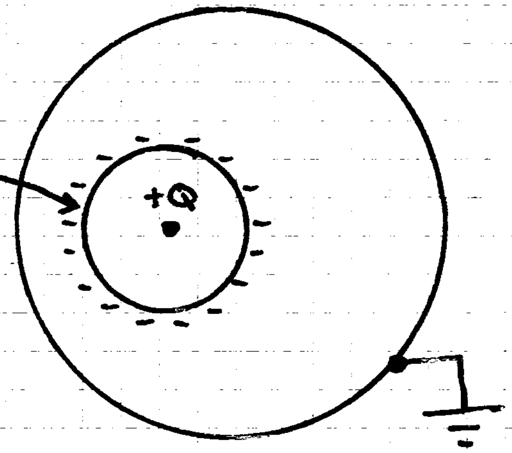
Outside metal

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\vec{a}_r$$

But now the origin is at the center of the metal sphere.

b)

uniform-
charge on inner
surface of cavity
total charge $-Q$



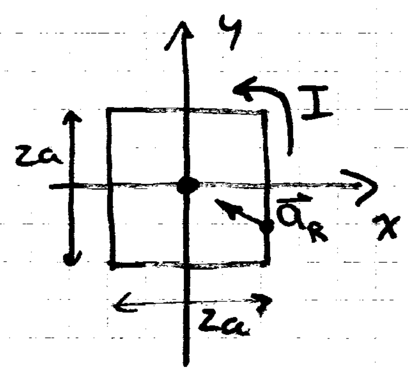
No other charge in metal

Inside cavity $\qquad \vec{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\,\vec{a}_r \qquad$ origin at Q

Outside metal $\qquad \vec{E} = 0$

Inside metal $\qquad \vec{E} = 0$

6.



$$\vec{H} = \oint \frac{I\, d\vec{l} \times \vec{a}_R}{4\pi R^2}$$

consider the segment at $x = +a$
the magnetic field from this segment
is

$$\vec{H}_1 = \int I \,\frac{d\vec{l} \times \vec{a}_R}{4\pi R^2} \qquad d\vec{l} = \vec{a}_y\, dy$$

$$\vec{a}_R = \frac{-a\,\vec{a}_x - y\,\vec{a}_y}{(a^2+y^2)^{1/2}}$$

so $\quad d\vec{l} \times \vec{a}_R = \dfrac{a\,\vec{a}_z}{(a^2+y^2)^{1/2}}\, dy$

and $\quad R = (a^2+y^2)^{1/2}$

so
$$\vec{H}_1 = \frac{I}{4\pi} \int_{-a}^{a} \frac{a\, dy}{(a^2+y^2)^{3/2}}\, \vec{a}_z$$

$$= \frac{Ia}{4\pi} \left[\frac{y}{a^2 (a^2+y^2)^{1/2}} \right]_{-a}^{a} \vec{a}_z$$

$$= \frac{Ia}{4\pi} \frac{2a\, \vec{a}_z}{a^2 (a^2+a^2)^{1/2}} = \frac{2I}{\sqrt{2}\, a\, 4\pi} \vec{a}_z$$

A moment's thought should convince you that the other three segments produce the same field vector at the center of the loop.

So
$$\vec{H} = \frac{2I}{\sqrt{2}\, a\, \pi} \vec{a}_z$$