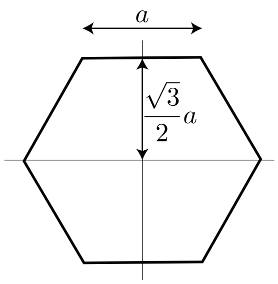
2012 EE301 Midterm II Solutions

1. Law of Biot-Savart

A wire is bent into the shape of a regular hexagon with side a as shown below. A current I flows in the wire. Determine the magnetic field vector at the centre of the hexagon (Think!).



You might be interested to know that

$$\int \frac{dx}{(x^2+c)^{3/2}} = \frac{x}{c(x^2+c)^{1/2}}$$

Answer:

Because of the symmetry of a regular hexagon, each side produces the same magnetic field vector at the centre. So we need calculate the vector for only one side and multiply the result by 6. The side at the top is a convenient choice. The vector is calculated using the law of Biot-Savart.

$$\vec{H} = \int \frac{I\,d\vec{l} \times \vec{a}_R}{4\pi R^2}$$

The direction of the current is not specified, so I arbitrarily choose the counter-clockwise direction; then $d\vec{l} = -dx\vec{a}_x$. $\vec{a}_R$ points from a bit of current at $(x, \sqrt{3}a/2)$ to the origin; so

$$\vec{a}_R = \frac{-x\vec{a}_x - (\sqrt{3}a/2)\vec{a}_y}{(x^2+3a^2/4)^{1/2}} \quad \text{and} \quad R = (x^2+3a^2/4)^{1/2}$$

Evaluating the cross product results in

$$\vec{H} = \int_{-a/2}^{a/2} \frac{I(\sqrt{3}a/2)\vec{a}_z}{4\pi(x^2+3a^2/4)^{3/2}}dx = \frac{\sqrt{3}aI}{8\pi}\int_{-a/2}^{a/2}\frac{dx}{(x^2+3a^2/4)^{3/2}}\vec{a}_z$$

$$\vec{H} = \frac{\sqrt{3}aI}{8\pi}\left[\frac{x}{(3a^2/4)(x^2+3a^2/4)^{1/2}}\right]_{-a/2}^{a/2} = \frac{I}{2\sqrt{3}\pi a}$$

Therefore, the total field vector is

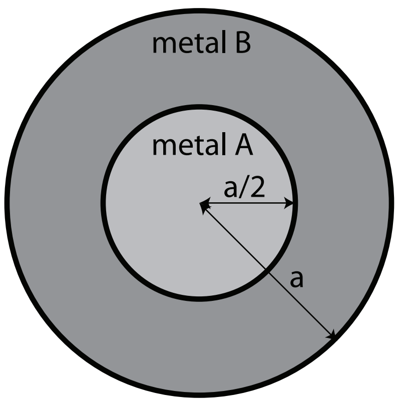
$$\vec{H}_{\text{total}} = \frac{\sqrt{3}I}{\pi a}$$

2. Ampere's law

A long, straight wire is constructed from two metals as shown below, metal A for $\rho < a/2$ and metal B for $a/2 < \rho < a$. A current I flows through the wire. Because of the difference in conductivity between the two metals, 90% of the current flows through metal B and only 10% through metal A.

a) The current density in each metal is constant. Write down the expression for the current density in each metal.

b) Use Ampere's law to determine an expression for the magnetic field in metal A, in metal B, and outside the wire.



Answer:

a) Current density is the current divided by the cross-sectional area.

$$\vec{j}_A = \frac{0.1I}{\pi(a/2)^2}\vec{a}_z = \frac{0.4I}{\pi a^2}\vec{a}_z \quad \text{and} \quad \vec{j}_B = \frac{0.9I}{\pi a^2 - \pi(a/2)^2}\vec{a}_z = \frac{1.2I}{\pi a^2}\vec{a}_z$$

b) The symmetries of the wire plus the cross product in the law of Biot-Savart require that the magnetic field be of the form $\vec{H}(\rho) = H_\phi \vec{a}_\phi$ (cylindrical coordinates). We choose a circular path of radius ρ centred on the wire for the integral in Ampere's law.

$$\oint_{\text{loop}} \vec{H} \cdot d\vec{l} = 2\pi\rho H_\phi = I_{\text{encl}}$$

The current through the loop depends on ρ. For $\rho > a$, all the current flows through the loop.

$$I_{\text{encl}} = I \quad \Rightarrow \quad H_\phi = \frac{I}{2\pi\rho} \quad \text{for } \rho > a$$

For $a/2 < \rho < a$, all the current in metal A flows through the loop plus a fraction of current from metal B.

$$I_{\text{encl}} = 0.1I + \frac{\pi\rho^2 - \pi(a/2)^2}{\pi a^2 - \pi(a/2)^2}(0.9I) = 0.1I + \left(\frac{4\rho^2}{3a^2} - \frac{1}{3}\right)(0.9I) = \left(\frac{1.2\rho^2}{a^2} - 0.2\right)I$$

$$H_\phi = \left(\frac{0.6\rho}{\pi a^2} - \frac{0.1}{\pi\rho}\right)I \quad \text{for } a/2 < \rho < a$$

For $\rho < a/2$, only a fraction of the current in metal A flows through the loop.

$$I_{\text{encl}} = \frac{\pi\rho^2}{\pi(a/2)^2}(0.1I) \quad \Rightarrow \quad H_\phi = \frac{0.2I\rho}{\pi a^2} \quad \text{for } \rho < a/2$$

3. EM waves

a) An electromagnetic plane wave travels in the $+z$ direction in free space. The average power density equals 0.1 W/m^2, the frequency is $f = 2$ GHz, and the polarization is circular. In the equation for the electric field of the electromagnetic wave

$$\vec{E} = \vec{E}_0 e^{j(\omega t - kz)}$$

determine $\vec{E}_0$, ω, and k.

b) A plane electromagnetic wave is normally incident from air onto a dielectric. Careful measurements show that the reflected wave has an amplitude that is 0.90 of the amplitude of the incident wave; the measurements do not determine the phase of the reflected wave. Determine the dielectric constant ϵ_R of the dielectric. Remember that for ordinary dielectrics, $\epsilon_R > 1$.

Answer:

a) A circularly polarized plane wave has $\vec{E}_0 = E_0(\vec{a}_x + e^{\pm j\pi/2}\vec{a}_y)$. The magnitude of the Poynting vector is

$$P = \frac{|\vec{E}_0|^2}{2\eta_0} = \frac{2E_0^2}{2\eta_0} = 0.1 \text{ W/m}^2 \quad \Rightarrow \quad E_0 = 6.1 \text{ V/m}$$

So

$$\vec{E}_0 = 6.1(\vec{a}_x + e^{\pm j\pi/2}\vec{a}_y)(\text{V/m})$$

Also

$$\omega = 2\pi f = 1.26 \times 10^{10} \text{ rad/s} \quad \text{and} \quad k = \omega/c = 41.9 \text{ m}^{-1}$$

b) The reflection coefficient from a dielectric gives both the relative amplitude and the phase of the reflected wave.

$$\Gamma = \frac{\eta - \eta_0}{\eta + \eta_0}$$

We are given that $|\Gamma| = 0.9$ so either $\Gamma = 0.9$ or $\Gamma = -0.9$. Solving for η for both cases gives

$$\Gamma = 0.9 \quad \Rightarrow \quad \eta = 7.16\text{K}\Omega \qquad \text{and} \qquad \Gamma = -0.9 \quad \Rightarrow \quad \eta = 19.8\Omega$$

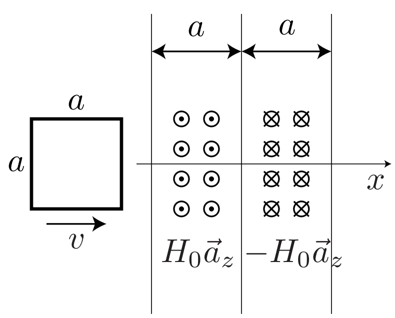
We should be immediately suspicious of the first value since it is greater than η_0. To convert η to dielectric constant, use $\eta = \sqrt{\mu_0/(\epsilon_0\epsilon_R)} = \eta_0/\sqrt{\epsilon_R}$. The first value of η above gives $\epsilon_R \ll 0$ which is clearly not correct. The other value gives $\epsilon_R = 361$.

4. Faraday's law

A square loop of wire with side a lies in the x-y plane and is moved at a constant velocity in the x direction. There is a magnetic field that in one region of width a is $\vec{H} = H_0\vec{a}_z$ and in an adjacent region also of width a is $\vec{H} = -H_0\vec{a}_z$, i.e. the field points in the opposite direction in the two regions but has a constant magnitude H_0.

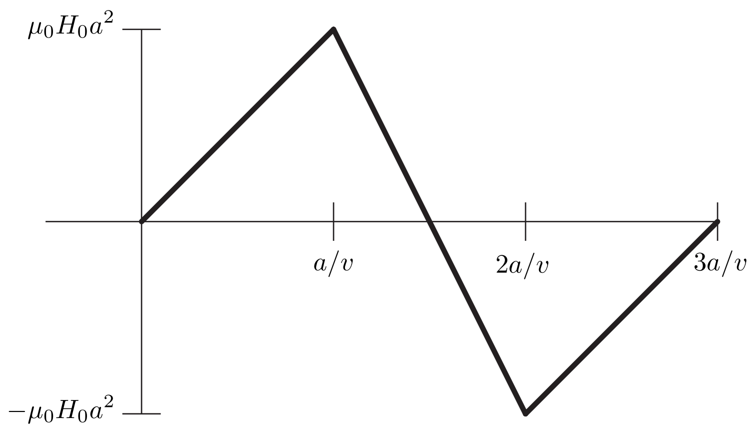
a) Draw an accurate graph of the magnetic flux through the loop as a function of time as the square loop passes through the two regions. Determine the maximum and minimum flux.

b) Draw an accurate graph of the emf induced in the loop as a function of time as the square loop passes through the two regions. Determine the maximum and minimum emf and indicate the direction of the induced current (clockwise or anti-clockwise) at the maximum and the minimum.



Answer:

a) Let the upward direction be positive. As the square loop enters the first region, the flux will increase linearly with time. The maximum is reached when the square is entirely inside the first region. The maximum flux is $\mu_0 H_0 a^2$. The time to reach the maximum flux is a/v. As the square moves into the second region, the flux decreases because the field is in the opposite direction. It reaches zero when the square is equally in both regions. The minimum flux, $-\mu_0 H_0 a^2$, occurs when the square is entirely in the second region. The time between maximum and minimum is a/v. The flux linearly returns to zero as the square leaves the second region in time a/v. A graph is shown below.



b) emf $= -d\phi/dt$. Since the flux always changes linearly with time, the emf is just the negative slope of the lines. The flux graph shows three parts. The emf in each part is

$$\text{emf}_1 = -\frac{\mu_0 H_0 a^2}{a/v} = -\mu_0 H_0 a v$$

$$\text{emf}_2 = \frac{2\mu_0 H_0 a^2}{a/v} = 2\mu_0 H_0 a v$$

$$\text{emf}_3 = -\frac{\mu_0 H_0 a^2}{a/v} = -\mu_0 H_0 a v$$

Since the induced current acts to oppose the change in flux, for the negative emf the current is clockwise and for the positive emf the current is anti-clockwise.