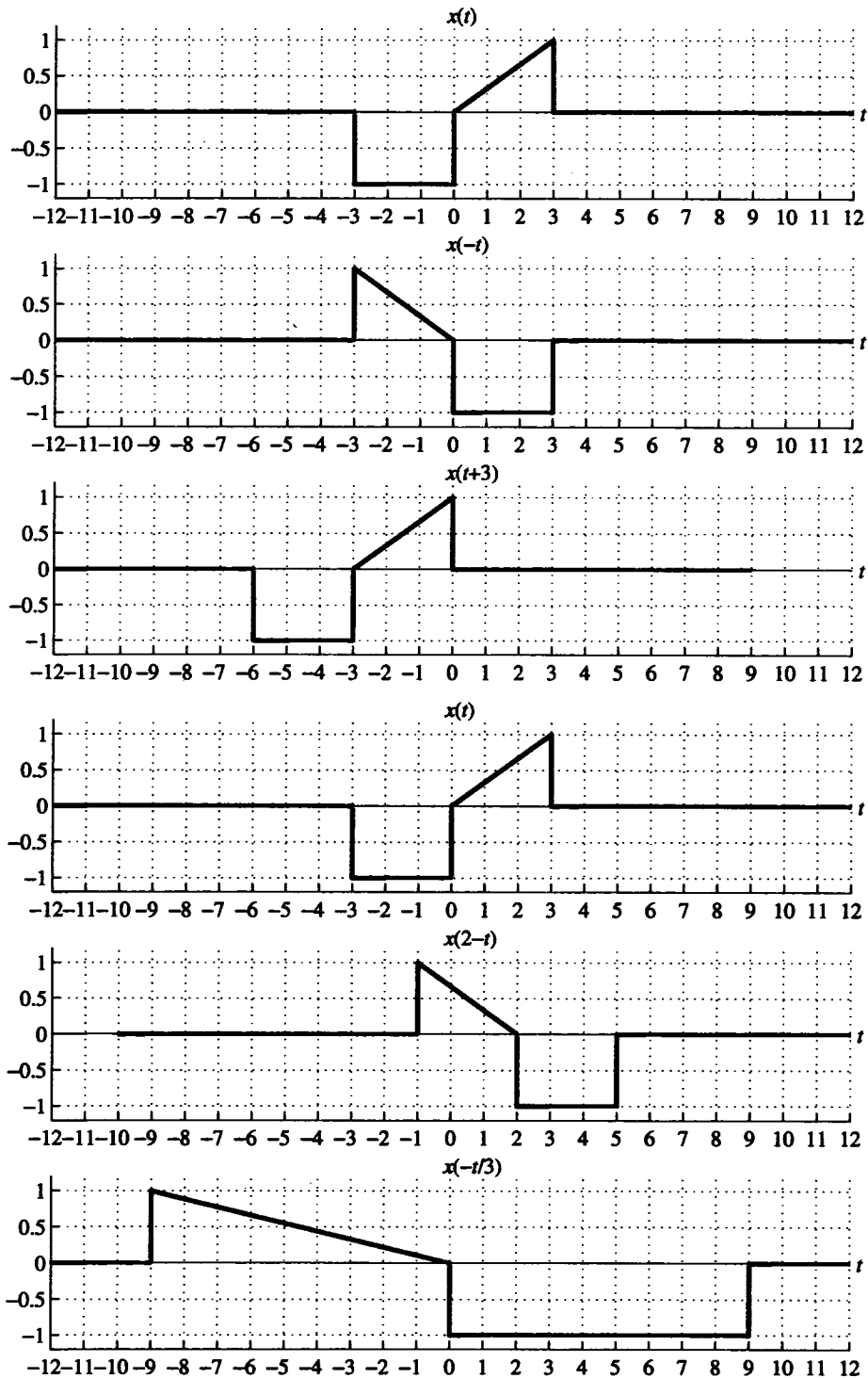


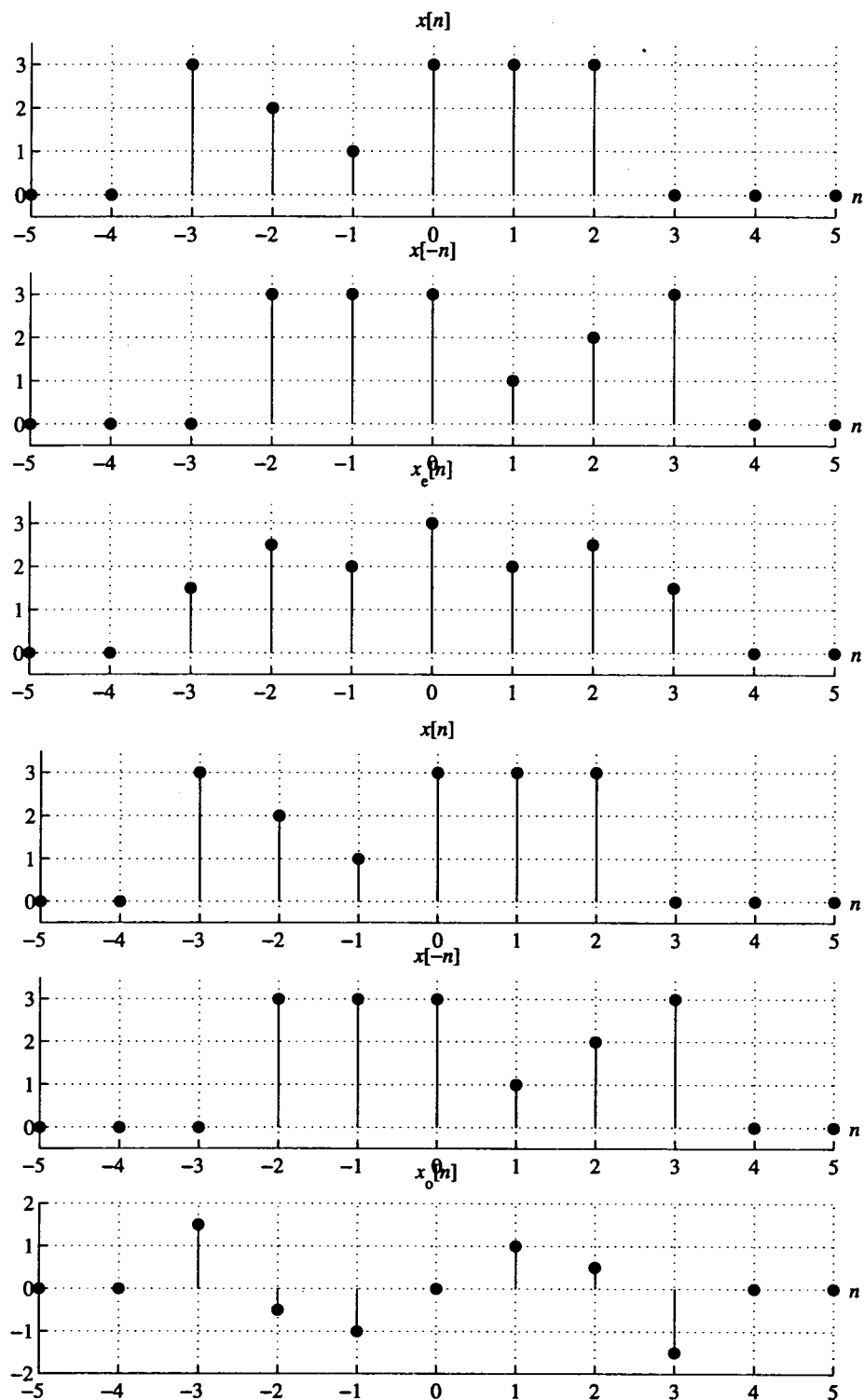
EE.351: Spectrum Analysis and Discrete-Time Systems
SOLUTIONS TO MIDTERM EXAM, Fall-2003

1. (Signal Transformations)

[6]

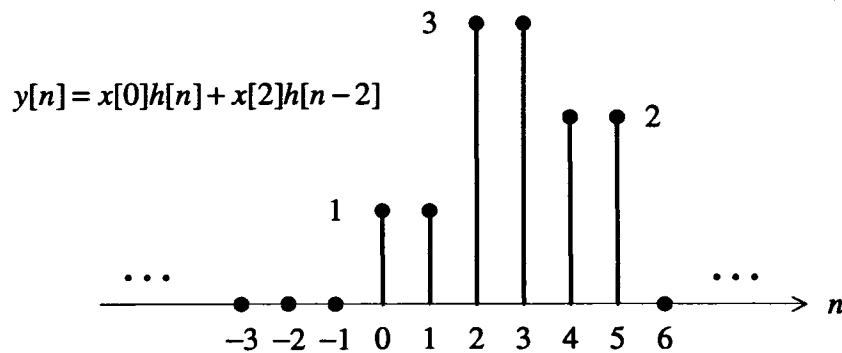
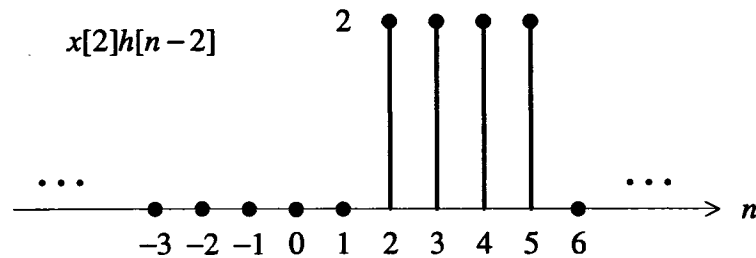
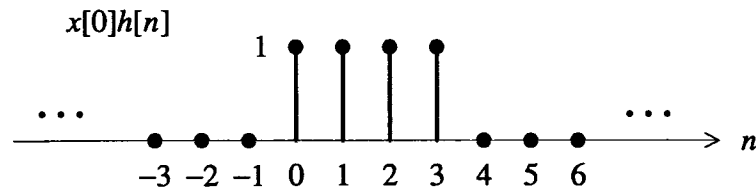
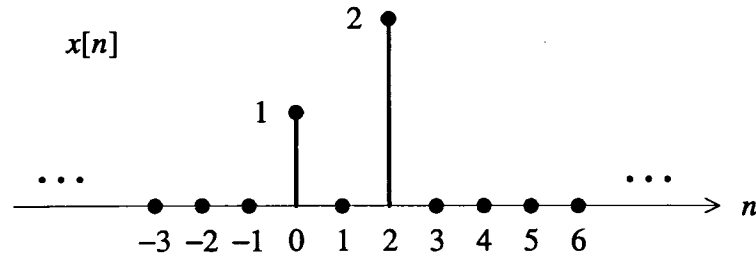
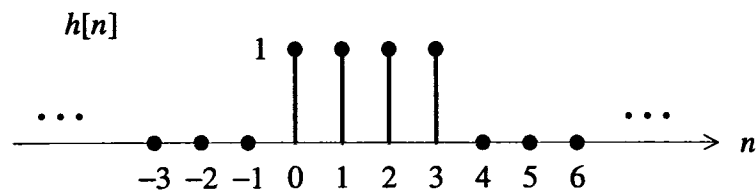


[4]

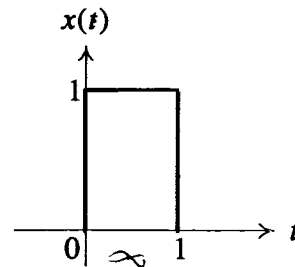
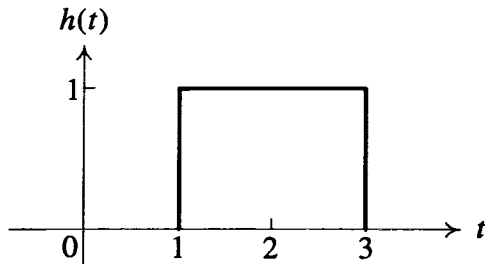


2. (Convolution)

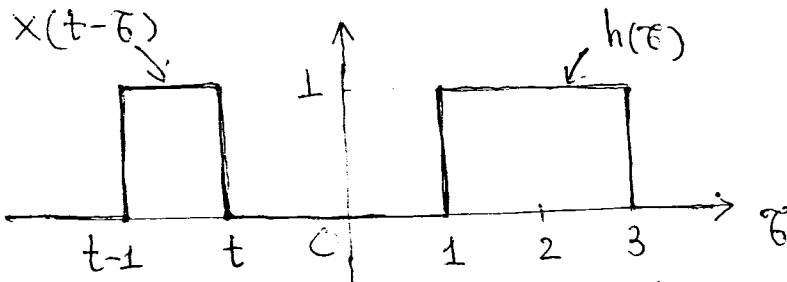
[5] (a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[0]h[n] + x[2]h[n-2]$.



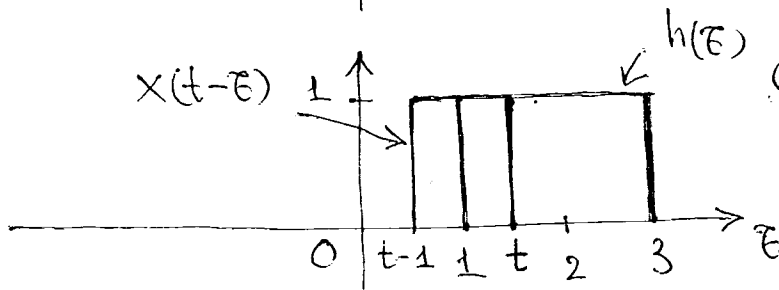
- (b) Consider a continuous-time LTI system with impulse response $h(t)$ and input $x(t)$ as shown below. Find and neatly sketch the output $y(t)$.



$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

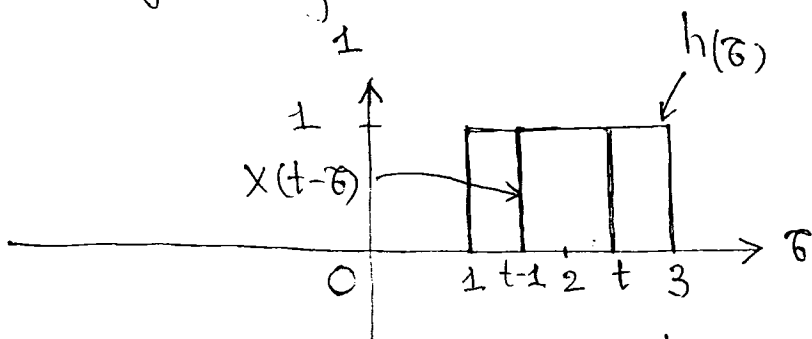


⊕ If $t \leq 1$ then
 $x(t-\tau) h(\tau) = 0 \Rightarrow y(t) = 0$



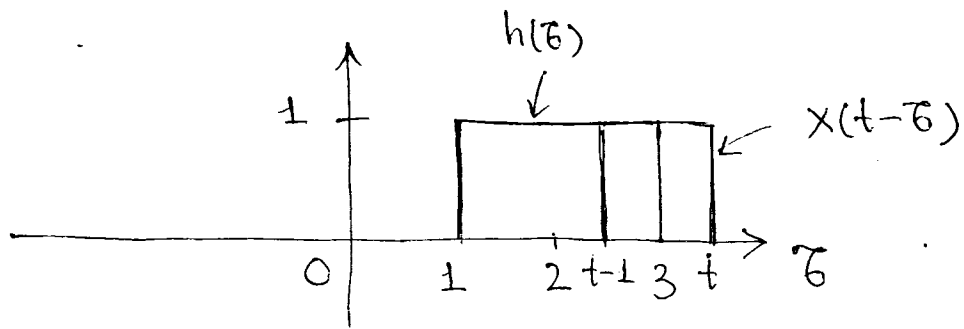
⊕ If $1 \leq t \leq 2$, then $x(t-\tau)$ and $h(\tau)$ are both non-zero over $1 \leq \tau \leq t$. Thus

$$y(t) = \int_{1}^{t} h(\tau) x(t-\tau) d\tau = \int_{1}^{t} 1 d\tau = t-1$$



⊕ If $2 \leq t \leq 3$, then both $h(\tau)$ and $x(t-\tau)$ are non-zero only in the interval $t-1 \leq \tau \leq t$.

Therefore:
$$y(t) = \int_{t-1}^{t} h(\tau) x(t-\tau) d\tau = \int_{t-1}^{t} 1 d\tau = 1$$

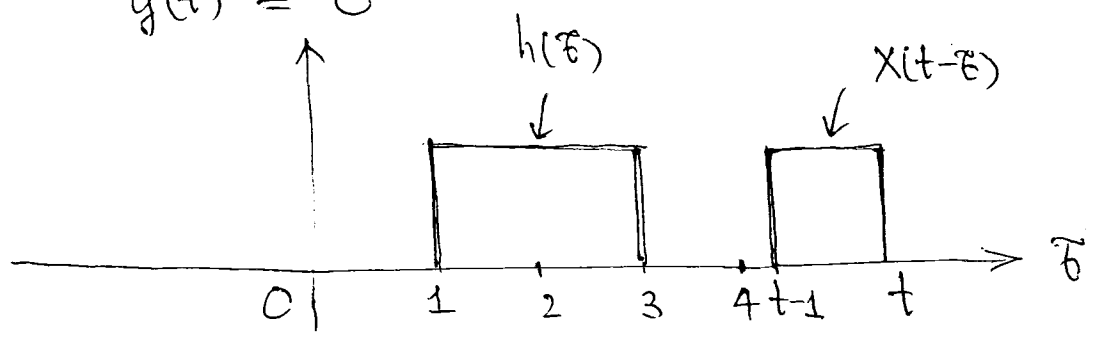


⊕ If $3 \leq t \leq 4$, then $h(\tau) x(t-\tau)$ is nonzero (equal to 1) over $t-1 \leq \tau \leq 3$. Therefore:

$$y(t) = \int_{t-1}^3 h(\tau) x(t-\tau) d\tau = \int_{t-1}^3 d\tau = 4-t$$

⊕ Finally, if $t > 4$, then $h(\tau)$ and $x(t-\tau)$ are non-overlapped $\Rightarrow h(\tau) x(t-\tau) = 0$. Hence

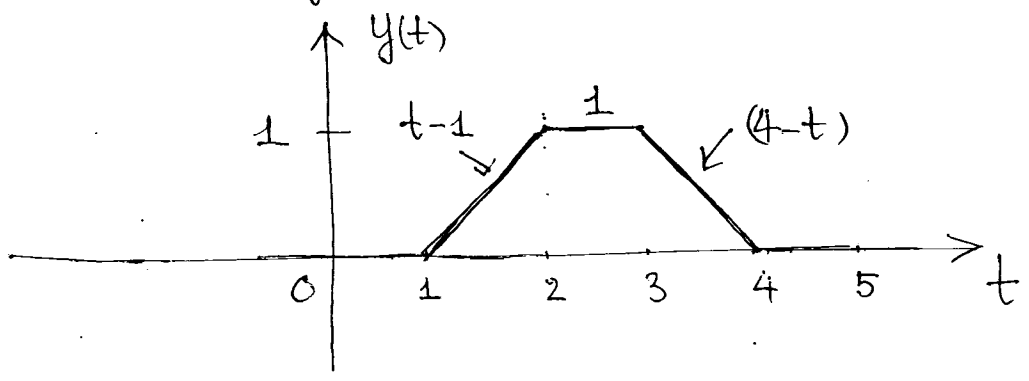
$$y(t) = 0$$



In summary:

$$y(t) = x(t) * h(t) = \begin{cases} 0 & ; t \leq 1 \text{ and } t \geq 4 \\ t-1 & ; 1 \leq t \leq 2 \\ 1 & ; 2 \leq t \leq 3 \\ 4-t & ; 3 \leq t \leq 4 \end{cases}$$

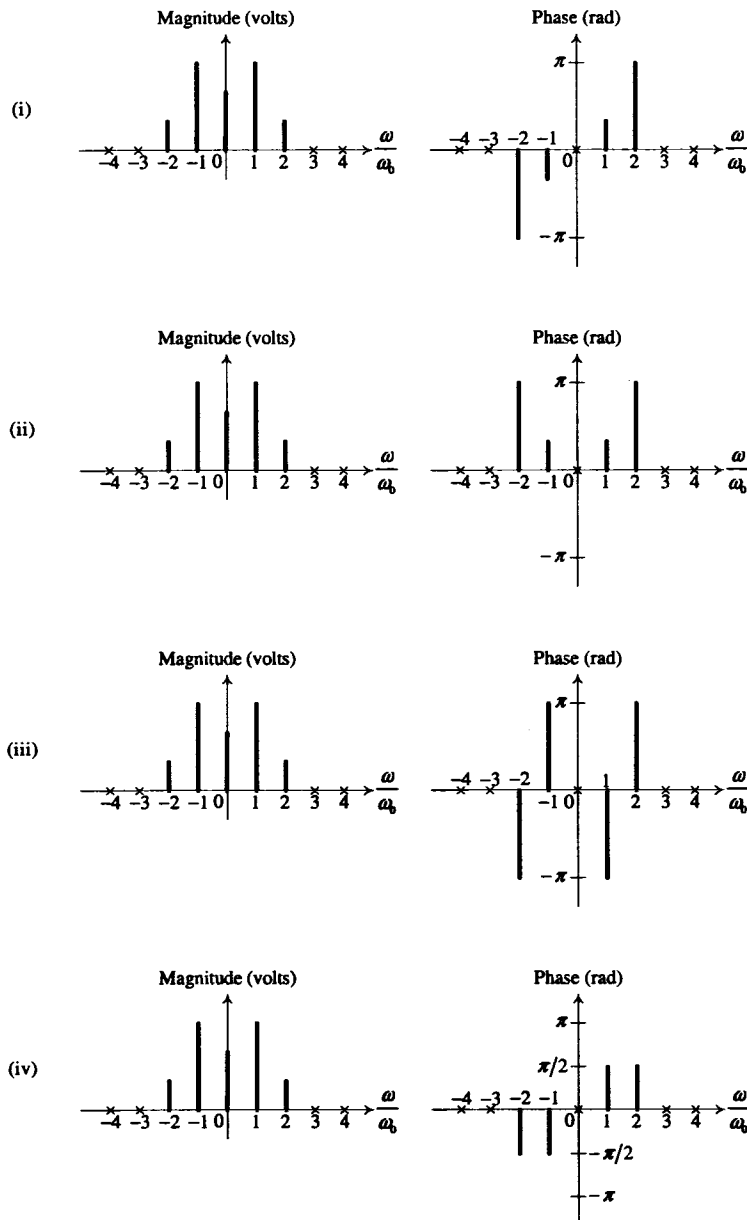
Sketch of $y(t)$ is shown below:



3. (Properties of Fourier Series Coefficients)

- [8] (a) Signals (i), (iii) and (iv) are real-valued since their magnitude spectra are *even* and phase spectra are *odd*. Signal (ii) is a complex-valued since its phase spectrum is *not* odd.
- [2] (b) Since $e^{j\pm\pi} = -1$ and $e^{j0} = 1$, the FS coefficients of signal (iii) are real-valued. Thus signal (iii) is both real-valued and even function.

None of the signal is both real-valued and odd since all the four signals have DC component. A real-valued odd signal must have a zero DC component, i.e., $a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) dt = 0$. *Remark:* If the DC component is removed, then the signal (iv) is both real-valued and odd because its FS coefficients are purely imaginary and odd ($e^{\pm j\pi/2} = \pm j$).



4. (Fourier Series Representation)

- [5] (a) A discrete-time periodic signal $x[n]$ is real-valued and has a fundamental period $N = 5$. The Fourier series coefficients for $x[n]$ are

$$a_0 = 2, \quad a_1 = a_{-1}^* = \frac{1}{2} - \frac{1}{2}j, \quad a_2 = a_{-2}^* = \frac{1}{2}j$$

- (i) Determine and neatly sketch the magnitude and phase spectra of $x[n]$ over at least two periods of a_k .

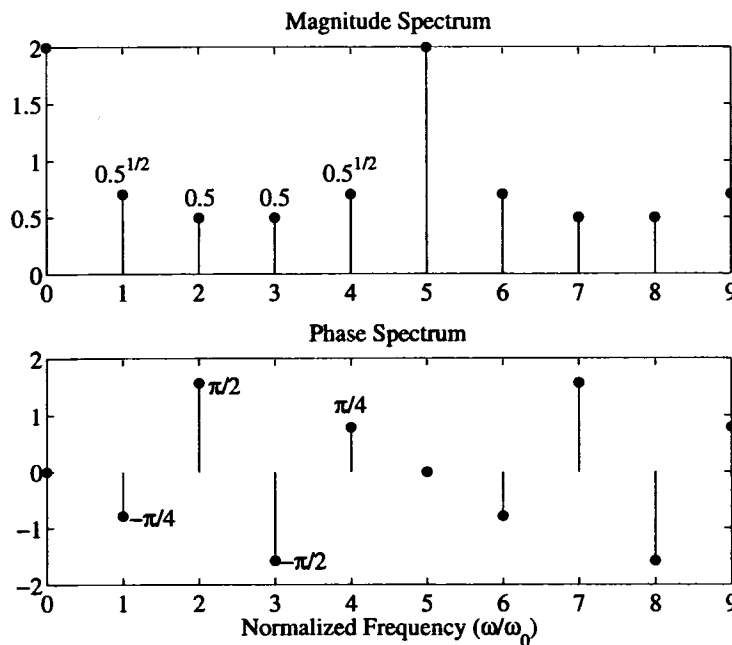
$$a_0 = 2 \rightarrow |a_0| = 2, \quad \angle a_0 = 0$$

$$a_1 = \frac{1}{2} - \frac{1}{2}j \rightarrow |a_1| = \frac{\sqrt{2}}{2}, \quad \angle a_1 = \arctan(-1) = -\pi/4$$

$$a_2 = \frac{1}{2}j = \frac{1}{2}e^{j\pi/2} \rightarrow |a_2| = \frac{1}{2}, \quad \angle a_2 = \pi/2$$

$$a_3 = a_{3-5} = a_{-2} = -\frac{1}{2}j = \frac{1}{2}e^{-j\pi/2} \rightarrow |a_3| = \frac{1}{2}, \quad \angle a_3 = -\pi/2$$

$$a_4 = a_{4-1} = a_{-1} = \frac{1}{2} + \frac{1}{2}j \rightarrow |a_4| = \frac{\sqrt{2}}{2}, \quad \angle a_4 = \arctan(1) = \pi/4$$

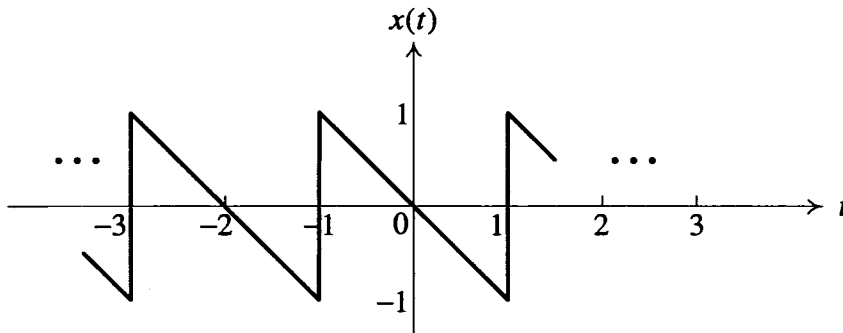


- (ii) Express $x[n]$ in the form: $x[n] = A_0 + \sum_{k=1}^{\infty} A_k \cos(\omega_k n + \phi_k)$.

The fundamental frequency is $\omega_0 = 2\pi/5$.

$$\begin{aligned}
 x[n] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=-2}^2 a_k e^{jk\omega_0 n} \\
 &= a_0 + [a_{-1}e^{-j\omega_0 n} + a_1e^{j\omega_0 n}] + [a_{-2}e^{-j2\omega_0 n} + a_2e^{j2\omega_0 n}] \\
 &= 2 + \left[\frac{\sqrt{2}}{2} e^{j\pi/4} e^{-j\omega_0 n} + \frac{\sqrt{2}}{2} e^{-j\pi/4} e^{j\omega_0 n} \right] + \left[\frac{1}{2} e^{-j\pi/2} e^{-j2\omega_0 n} + \frac{1}{2} e^{j\pi/2} e^{j2\omega_0 n} \right] \\
 &= 2 + \frac{\sqrt{2}}{2} [e^{-j(\omega_0 n - \pi/4)} + e^{j(\omega_0 n - \pi/4)}] + \frac{1}{2} [e^{-j(2\omega_0 n + \pi/2)} + e^{j(2\omega_0 n + \pi/2)}] \\
 &= 2 + \sqrt{2} \cos\left(\omega_0 n - \frac{\pi}{4}\right) + \cos\left(2\omega_0 n + \frac{\pi}{2}\right) \\
 &= 2 + \sqrt{2} \cos\left(\frac{2\pi}{5}n - \frac{\pi}{4}\right) + \cos\left(\frac{4\pi}{5}n + \frac{\pi}{2}\right)
 \end{aligned}$$

- [5] (b) Consider the following continuous-time periodic signal $x(t)$



- (i) The fund. period is $T_0 = 2$ and the fund. frequency is $\omega_0 = \frac{2\pi}{T_0} = \pi$.
(ii) To compute the Fourier series coefficients a_k of $x(t)$, consider $x(t)$ in one period, from $-1 \leq t \leq 1$. Then $x(t) = -t$, $-1 \leq t \leq 1$. Furthermore, observe that $x(t)$ is a real-valued, odd function. Thus the FS coefficients are purely imaginary. Hence $B_k = 0$ and $a_k = jC_k$, where

$$\begin{aligned}
 C_k &= -\frac{1}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t) dt = \frac{1}{2} \int_{-1}^1 t \sin(\omega_0 k t) dt \\
 &= \frac{1}{2} \left[\frac{\sin(\omega_0 k t)}{(\omega_0 k)^2} - \frac{t}{\omega_0 k} \cos(\omega_0 k t) \right] \Big|_{-1}^1 = -\frac{1}{2\pi k} [\cos(k\pi) + \cos(-k\pi)] \\
 &= -\frac{\cos(k\pi)}{\pi k} = -\frac{(-1)^k}{\pi k}
 \end{aligned}$$

The DC component of $x(t)$ is 0 since $x(t)$ is a real-valued, odd function.
To conclude

$$a_k = \begin{cases} 0, & k = 0 \\ -j \frac{(-1)^k}{k\pi}, & k = \pm 1, \pm 2, \dots \end{cases}$$