Engineering Physics 214 Midterm

There are five questions worth a total of 44 marks
No calculators or notes are allowed, a formula sheet is provided.
Question parts are equally weighted unless otherwise specified.

Tuesday March 3rd, 2015

1. (8 marks) Answer the following questions assuming $z_1 = -1 + j\sqrt{3}$ and $z_2 = \sqrt{2}e^{j\frac{\pi}{4}}$.
   (a) Express z_1 in the form of $Me^{j\phi}$
   (b) Express z_2 in the form of $x + jy$
   (c) What is $z_1 z_2$?
   (d) What is $\frac{z_1}{z_2}$?
   (e) At what frequency does $e^{z_1 t}$ oscillate?

2. (4 marks) Derive (through integration) the Laplace transform of:

$$f(t) = \sin(at),$$

   assuming $f(t)$ begins at time $t = 0$.

3. (12 marks) Consider the series RLC circuit shown in Figure 1 with arbitrary input voltage and a switch that closes at $t = 0$.

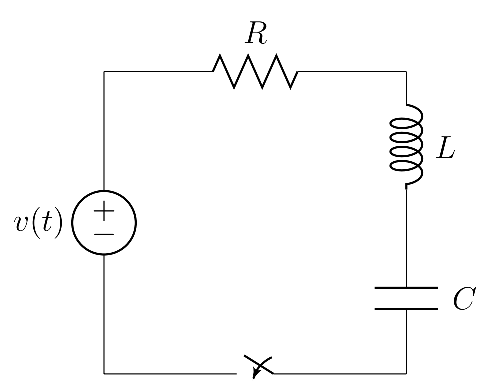


Figure 1: RLC circuit.

   (a) What is the differential equation that describes the current in the circuit?
   (b) Derive the transfer function for the current.
   (c) Derive the unit impulse response for the current assuming the system is critically damped.
   (d) If the circuit is driven by a voltage $v(t) = \tilde{A}e^{j\omega t}$ what is the Laplace transform of the current, $I(s)$, assuming all initial conditions are zero?

4. (14 marks) Consider the series RC circuit shown in Figure 2 where the switch closes at $t = 0$.

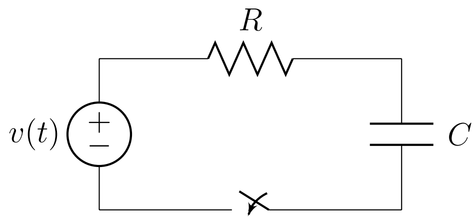


Figure 2: RC circuit.

(a) (4 marks) Derive the zero input response of the capacitor voltage assuming an initial charge of $q(0) = q_0$.

(b) (4 marks) Derive the transfer function for the voltage across the capacitor.

(c) (4 marks) If the circuit is driven by a constant voltage, $v(t) = V_0$, derive the capacitor voltage as a function of time.

(d) (2 marks) If the circuit is driven by the voltage, $v(t) = \frac{1}{t}\sin(\omega t)u(t)$, what is the capacitor voltage as a function of time. State your answer in the form of an integral but do not evaluate it.

5. (6 marks) Consider a damped mass-spring system described by the equation

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Where $k = 2\,\mathrm{N/m}$ and $m = 8\,\mathrm{kg}$.

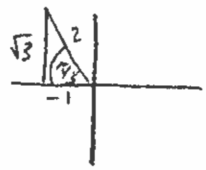
(a) What is the damping factor, b, that results in a critically damped system?

(b) If the system is underdamped sketch the position of the mass as a function of time if $\dot{x}(0) = 0\,\mathrm{m/s}$ and $x(0) = 1\mathrm{m}$. On this same plot sketch an example of a critically damped case with the same initial conditions.

End of Exam

PROBLEMS | CLASS | NAME | DATE

1) $z_1 = -1 + j\sqrt{3}$



$z_1 = 2e^{j2\pi/3}$

b) $z_2 = \sqrt{2}\, e^{j\pi/4}$

$z_2 = 1 + j$

c) $z_1 z_2 = M_1 M_2 e^{j(\phi_1 + \phi_2)}$

$= 2\sqrt{2}\, e^{j(\frac{2\pi}{3} + \pi/4)}$

$= 2\sqrt{2}\, e^{j(11\pi/12)}$

d) $z_1/z_2 = M_1/M_2\, e^{j(\phi_1 - \phi_2)}$

$= 2/\sqrt{2}\, e^{j(2\pi/3 - \pi/4)}$

$= \sqrt{2} e^{j(5\pi/12)}$

e) $e^{z_1 t} = e^{t(-1+j\sqrt{3})} = e^{-t} e^{j\sqrt{3}t} = e^{-t}\left(\cos(\sqrt{3}t) + j\sin(\sqrt{3}t)\right)$

$\omega = \sqrt{3} \rightarrow f = \sqrt{3}/2\pi$

2) $F(s) = \int_0^\infty \sin(at) e^{-st}\, dt = \frac{1}{2j}\int_0^\infty (e^{jat} - e^{-jat}) e^{-st}\, dt$

$= \frac{1}{2j}\int_0^\infty \left(e^{(-s+ja)t} - e^{(-s-ja)t}\right) dt$

$= \frac{1}{2j}\left[\frac{1}{-s+ja} e^{(-s+ja)t} - \frac{1}{-s-ja} e^{(-s-ja)t}\right]_0^\infty$

$= \frac{1}{2j}\left[\frac{1}{-s+ja}(0-1) - \frac{1}{-s-ja}(0-1)\right]$

$= \frac{1}{2j}\left[\frac{1}{-s-ja} - \frac{1}{-s+ja}\right]$

$= \frac{1}{2j}\left[\frac{-s+ja + s + ja}{(-s-ja)(-s+ja)}\right] = \frac{1}{2j}\frac{2ja}{s^2+a^2}$

$= \frac{a}{s^2+a^2}$

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3a) $v_L + v_C + v_R = v$

$$\frac{di}{dt}L + \frac{1}{C}\int_0^t i(\tau)\,d\tau + \frac{q_0}{C} + iR = v$$

b) $$sI(s)L + \frac{1}{sC}I(s) + \underbrace{\frac{q_0}{sC}}_{q_0=0} + I(s)R = V(s)$$

$$I(s)\left(sL + R + \frac{1}{sC}\right) = V(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{sL + R + \frac{1}{sC}} = \frac{s}{s^2L + sR + 1/C}$$

c) $h(t) = \mathcal{L}^{-1}(H(s))$

$$H(s) = \frac{s/L}{(s-s_1)(s-s_2)} = \frac{A}{(s-s_1)} + \frac{B}{(s-s_1)^2}$$ since $s_1 = s_2$ we have a double root

OR

$$h(t) = \frac{1}{L}\frac{d}{dt}\left(te^{s_1t}\right)$$

from multiplication by s in the frequency domain

$$h(t) = \frac{1}{L}\left(s_1te^{s_1t} + e^{s_1t}\right)$$

d) $v(t) = \tilde{A}e^{j\omega t} \rightarrow V(s) = \dfrac{\tilde{A}}{s - j\omega}$

$$I(s) = H(s)V(s) = \frac{\tilde{A}}{s - j\omega}\,\frac{s}{(s-s_1)^2}\cdot\frac{1}{L}$$

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4 a) $v = v_R + v_c$

$v = \dot{q}R + q/C$

$V = sQR - q(0)R + Q/C$

$V = Q(sR + \frac{1}{C}) - q_0 R$ Zero input when $V = 0$

$$Q(s) = \frac{q_0 R}{sR + 1/C}$$

$$= q_0 R\left(\frac{1}{sR + 1/C}\right) = q_0 R\left(\frac{1/R}{s + 1/RC}\right)$$

$$q(t) = q_0 e^{-t/RC} \rightarrow v_c(t) = \frac{q_0}{C} e^{-t/RC}$$

b) $V(s) = Q(s)\,(sR + \frac{1}{C})$

$$\frac{Q(s)}{V(s)} = \frac{1}{sR + \frac{1}{C}} = \frac{1}{R}\,\frac{1}{s + 1/RC}$$

$$H_Q(s) = \frac{1}{R}\,\frac{1}{s + \frac{1}{RC}} \rightarrow H_{V_c} = \frac{1}{RC}\,\frac{1}{s + \frac{1}{RC}}$$

c) $v(t) = V_0 \longrightarrow V(s) = \frac{V_0}{s}$

$$V_c(s) = V(s)\,H_c(s) = \frac{V_0}{RCs}\,\frac{1}{s + 1/RC}$$

$$v_c(t) = \frac{1}{s_1 - s_2}\left(e^{s_1 t} - e^{s_2 t}\right) = \frac{V_0}{RC}\,RC\left(e^{0t} - e^{-t/RC}\right)$$

$$= V_0\left(1 - e^{-t/RC}\right) + \frac{q_0}{C} e^{-t/RC}$$

d) $v_c(t) = h(t) * v(t) + ZIR$

$$= \underbrace{\int_0^t \frac{1}{t}\sin(\omega t)\,\frac{1}{RC} e^{-t/RC}\,dt}_{ZSR} + \underbrace{q_0 e^{-t/RC}/C}_{ZIR}$$

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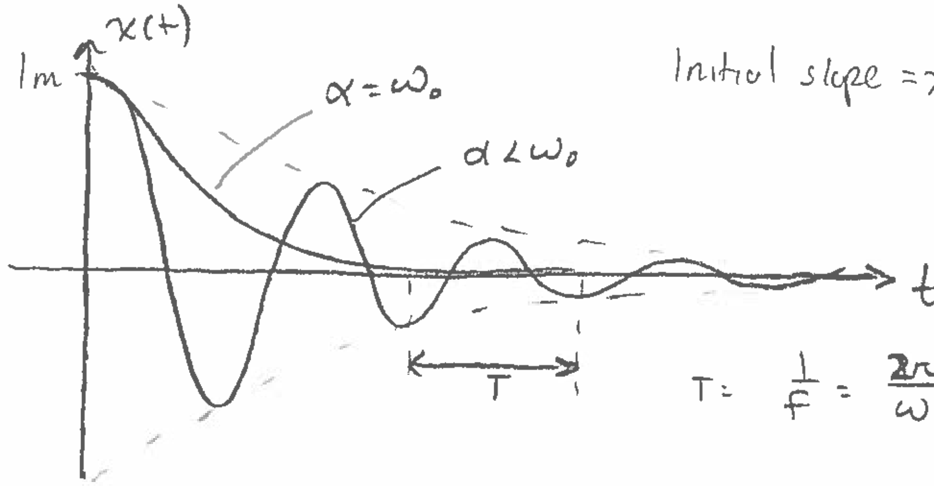
5) a) let $\alpha = \frac{b}{2m}$ $\omega_0 = \sqrt{\frac{k}{m}}$

$$\frac{b}{2m} = \sqrt{\frac{k}{m}} \rightarrow b = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km}$$

$$b = 2\sqrt{2 \cdot 8} = 8 \text{ kg/s}$$

b)



Initial slope $= \dot{x}(0) = 0$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \alpha^2}}$$