EP 214 - 2016 Midterm - Question 1 Solution

a) $z(t) = 3.2\, e^{-j(\omega t + \theta)}$

$\omega = 2\pi f$
$f = 100\,Hz$
$\theta = \pi/7$
$t = 2\,ms$

$\omega = 200\pi$ rad/s
$\omega t = (200\pi)(0.002) = 0.4\pi$

$\omega t + \theta = 1.7054.$

$z(t) = 3.2\, e^{-j1.7054}$ ⟸ not between zero and 2π

$\boxed{z(t) = 3.2\, e^{j4.58}}$ ← θ between $[0, 2\pi]$.

b) $\dot{z}(t) = -j\omega\, 3.2\, e^{-j(\omega t + \theta)}$

$\theta = \pi/2 \qquad \omega t = \pi/2$

$f = 100\,Hz$
$t = 0.0025\ s$
$\theta = \pi/2$

$\omega = 200\pi$

$\dot{z}(t) = -j(200\pi)(3.2)\, e^{-j\pi}$

$= 640\pi j = \boxed{640\pi\, e^{j\pi/2} = \dot{z}(t)}$

Question 1

c)

$$Z(\omega) = \frac{Z_c}{Z_c + Z_R}$$

$$Z_R = R = 10\Omega$$

$$Z_c = \frac{-j}{\omega C} = -j\,10\,\Omega$$

$$\omega = 1\times10^5 \ \frac{rad}{s}$$

$$C = 1\,\mu F$$

$$Z(\omega) = \frac{-j\,10\Omega}{10\Omega - j10\Omega} = \frac{10}{10}\left(\frac{-j}{1-j}\right) = \frac{-j\,(1+j)}{2}$$

$$\boxed{Z(\omega) = \frac{1-j}{2} = 0.7071\, e^{j\,5.4978}}$$

Question 2 solutions

a) $$\boxed{\begin{array}{l} \ddot{q} + 2\alpha\dot{q} + \beta q = \dfrac{v(t)}{L} \\ \alpha = \dfrac{R}{2L} \qquad \beta = \dfrac{1}{LC} \end{array}}$$

b) Use

$$L\frac{di}{dt} + iR + \frac{q(0)}{C} + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t).$$

$$LsI(s) + I(s)R + \frac{q(0)}{C}\frac{1}{s} + \frac{I(s)}{sC} = V(s)$$

$$I(s)\left[Ls + R + 1/sC\right] = V(s) - \frac{q(0)}{C}\frac{1}{s}$$

$$\boxed{I(s) = \frac{V(s)}{sL + R + 1/sC} - \frac{q(0)}{C}\frac{1}{s}\frac{1}{sL+R+1/sC}}$$

or $$\boxed{\frac{sV(s)}{L}\frac{1}{s^2+2\alpha s+\beta} - \frac{\beta q(0)}{s^2+2\alpha s+\beta}}$$

c) Use $$\ddot{q} + 2\alpha\dot{q} + \beta q = \frac{v(t)}{L}$$

$q(0) = 0$ as we are looking for unit impulse response.

$$s^2Q(s) + 2\alpha sQ(s) + \beta Q(s) = \frac{V(s)}{L}$$

but $V(s) = 1$

so

$$Q(s) = H(s) = \frac{1}{L} \frac{1}{s^2 + 2\alpha s + \beta}$$

$$= \frac{1}{L} \frac{1}{(s - s_1)} \frac{1}{(s - s_2)}$$

so

$$\boxed{h(t) = \frac{1}{L} \frac{1}{(s_1 - s_2)} \left[e^{s_1 t} - e^{s_2 t} \right]}$$

from tables of transforms

d) For no initial conditions.

$$I(s) = \frac{V(s)}{sL + R + 1/sC}$$

for Transter functions $V(s) = 1$

$$H_I(s) = \frac{1}{sL + R + 1/sC}$$

since $V_L(s) = sL I(s)$ for zero initial conditions

$$V_L(s) = sL \frac{V(s)}{sL + R + 1/sC}$$

and $$\boxed{H_{V_L}(s) = \frac{sL}{sL + R + 1/sC}}$$

e) Use $\ddot{q} + 2\alpha\dot{q} + \beta q = \frac{v(t)}{L}$

but $q(0) \neq 0$ and $\dot{q}(0) = 0$

so

$$\frac{V(s)}{L} = s^2 Q(s) - s q(0) + 2\alpha[sQ(s) - q(0)] + \beta Q(s)$$

with $V(s) = A \frac{s}{\omega^2 + s^2}$.

so $$Q(s) = \frac{V(s)}{L(s^2 + 2\alpha s + \beta)} + \frac{s q(0) + 2\alpha q(0)}{(s^2 + 2\alpha s + \beta)}$$

$$\boxed{Q(s) = \frac{A}{L} \frac{s}{\omega^2 + s^2} \frac{1}{(s^2 + 2\alpha s + \beta)} + \frac{s q(0) + 2\alpha q(0)}{(s^2 + 2\alpha s + \beta)}}$$

Question 3

①

a) $\boxed{L\frac{di}{dt} + iR = v(t)}$

b) $i(0) = 0$ $\quad sL\,I(s) + RI(s) = 1$ ↙ transfer function

$$\boxed{I(s) = \frac{1}{R+sL} = H(s)}$$

c) $H(s) = \frac{1}{L}\,\frac{1}{s+R/L}$ $\quad$ so $\boxed{h(t) = \frac{1}{L}e^{-R/L\,t}}$

d) $i(t) = h(t) * v(t)$

$$\boxed{i(t) = \int_0^{\infty} v(t-\tau)\,\frac{1}{L}e^{-R/L\,\tau}\,d\tau}$$

e) $I(s) = \frac{V(s)}{R+sL}$ $\quad V(s) = \frac{\tilde{A}}{s-j\omega}$

so $\boxed{I(s) = \frac{\tilde{A}}{L}\,\frac{1}{(s-j\omega)(s+R/L)}}$

Question 4.

a) $X(s) = \frac{s}{s^2+100} - \frac{4}{s(s+2)} + 7.$

$$x(t) = \cos(10t) - 4\int_0^t e^{-2\tau}d\tau + 7\delta(t)$$

$$= \cos(10t) + 2e^{-2t} - 2 + 7\delta(t)$$

b) $L = 2mH$ $\quad C = 15\mu F$

$$\alpha = \frac{R}{2L} \qquad \beta = \frac{1}{LC} = \frac{1}{(2\times10^{-3})(15\times10^{-6})} = 3.333\times10^{7}$$

for underdamped.

$$\alpha^2 < \beta \qquad \text{or} \quad \frac{R^2}{4L^2} < \frac{1}{LC}$$

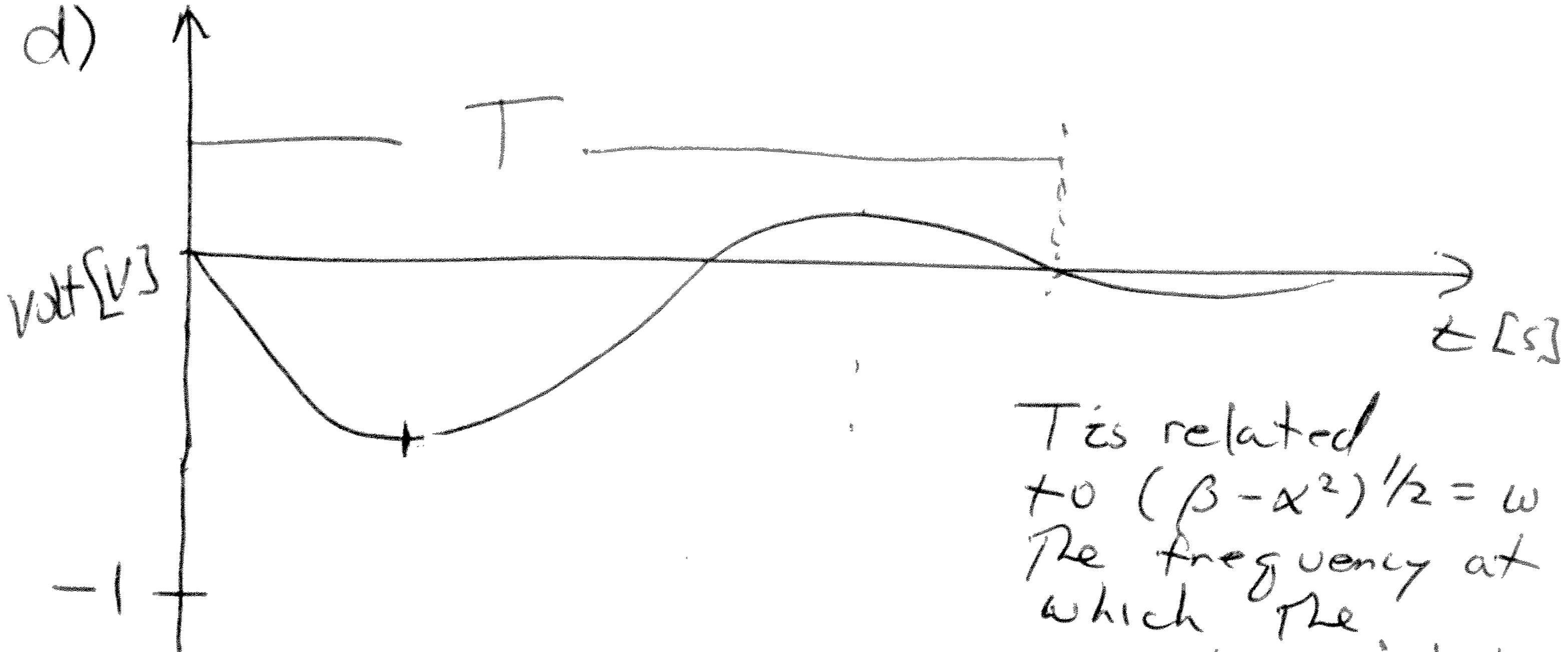
$$R^2 < \frac{4L}{C}$$

$$\boxed{R < 2\sqrt{\frac{L}{C}} = 2\left(\frac{2\times10^{-3}}{15\times10^{-6}}\right)^{1/2} = 23.1\ \Omega}$$

c) for $x(t) = u(t-2) + 4\sin(\omega t) - 8t$

$$\boxed{X(s) = \frac{e^{-2s}}{s} + \frac{4\omega}{s^2+\omega^2} - \frac{8}{s^2}.}$$

d)



T is related to $(\beta-\alpha^2)^{1/2} = \omega$ the frequency at which the circuit oscillates.

e) $\ddot{q} + 2\alpha\dot{q} + \beta q = v(t)$ for the charge on capacitor.

$$L\frac{di}{dt} + iR + \frac{q(0)}{C} + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t)$$

for the current.

$i(t) = \dot{q}(t)$ for the current.

$v_{RC}(t) = iR + q/C$ for the voltage drop across components.